Workshop in symplectic geometry:
Lefschetz fibrations: rigidity and flexibility
January 18th to 24th, 2016

Thursday Jan. 21, 2016:

Friday Jan. 22, 2016:

Saturday Jan. 23, 2016:

Notes taken by Cédric De Groote (cedricd@stanford.edu)
The workshop took place from Monday Jan. 18, 2016 to Sunday Jan. 24, 2016 at the Kerlerec House (928 Kerlerec Street, New Orleans, LA 70116).

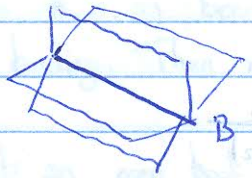
Morgan Weiler - Open books

Definition: an abstract open book is a manifold Σ with nonempty boundary and a diffeo ϕ of Σ wich is id on a collar of $\partial\Sigma$

An open book decomposition of a closed n-manifold N is a codim 2 submanifold B^{n-2} (the binding) and a smooth locally trivial fibration $\phi: N \setminus B \to S^1$. The pages are $\phi^{-1}(\theta)$. And B has a trivial tubular neighbourhood $B \times D^2$ on which $p(b, re^{i\theta}) = [\theta]$.

Note: $\overline{p^{-1}(\theta)}$ is a codim 1 submanifold with $\partial = B$.

ex in $\mathbb{R}^3$:



How do we get from one to the other?

$\boxed{\text{AOB} \Rightarrow \text{OBD:}}$ take the mapping torus $\Sigma(\phi) = \Sigma \times [0, 2\pi] / (x, 2\pi) \sim (\phi(x), 0)$; we have $\partial\Sigma(\phi) = \partial\Sigma \times S^1$, because $\phi = \text{id}$ near $\partial\Sigma$. Glue in $\partial\Sigma \times D^2$ via id. And we have $p([x, \theta]) = [\theta]$

$p(t, re^{i\theta}) = [\theta]$ (t is the coordinate on $\partial\Sigma$)

$\boxed{\text{OBD} \Rightarrow \text{AOB:}}$ set $\Sigma := p^{-1}(0) \cap (N \setminus B \times \mathring{D}^2_{1/2})$ ("take away the binding and chose a page") Choose a metric on N such that ∂_θ on $B \times (D^2 \setminus \{0\})$ is orthogonal to the pages. Extend ∂_θ to $\tilde{\partial}_\theta$ defined on all N^B which is orthogonal to the pages and $p_*(\tilde{\partial}_\theta) = \partial_\theta$. Rescale $\tilde{\partial}_\theta$ near the binding to get a vector field zero on B. Take the time-(2π) flow to get ϕ.

Rem: we had to choose a metric.

To check:

1) OBD$(B, p) \rightsquigarrow$ AOB$(\Sigma_{(B,p)}, \phi_{(B,p)}) \rightsquigarrow$ OBD$(B'_{\Sigma(B,p)}, p'_{\phi(B,p)})$ are diffeo.

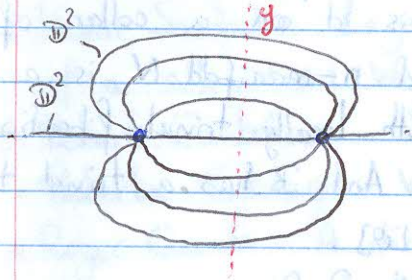
2) AOB$(\Sigma, \phi) \rightsquigarrow$ OBD $\rightsquigarrow$ AOB gives $\Sigma_{(B(\Sigma,\phi), p(\Sigma,\phi))}$ diffeo to Σ, and the maps ϕ and $\phi_{(B(\Sigma,\phi), p(\Sigma,\phi))}$ differ by a change of coordinates.
This is what we call equivalent AOBs.

3) Equivalent AOBs give diffeo manifolds.

Note: there are many "different" OBD of the same manifold.

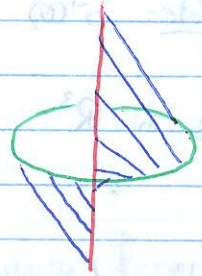
Theorem [Alexander]: a closed connected 3-manifold M admits an OBD with connected binding.

Examples: OBD of S^3:

1) 

Rotate around the y axis.
Binding: S^1
Pages: $\mathbb{D}^2$
The AOB is $(\mathbb{D}^2, id)$

2) $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$
$B = \{(z_1, z_2) \mid z_1 z_2 = 0\}$ = red + green
$p : S^3 \setminus B \to S^1 : (z_1, z_2) \mapsto \frac{z_1 z_2}{|z_1 z_2|}$
Page = blue + rotates around green



We see that for AOB, $\Sigma = A$ and ϕ is a Dehn twist around , since we rotated around the green one.

Theorem [Giroux]: let M^3 be closed oriented.
$$\left\{\begin{matrix}\text{oriented contact}\\ \text{structures } \xi \text{ on } M\end{matrix}\right\} \Big/ \begin{matrix}\text{isotopy}\\ \text{through contact}\end{matrix} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{OBD of } M\\ \text{supporting } \xi\end{matrix}\right\} \Big/ \begin{matrix}\text{positive}\\ \text{stabilization}\end{matrix}$$

Positive stabilization: add 1-handle along boundary, and Dehn twist. This is what happened from ex 1) to ex 2) above.

Definition: M^n, n odd, with OBD (B,p), B oriented, pages oriented by "induced orientation on $\partial\,\overline{page}$ = orientation of B" ($\overset{B \text{ odd dim}}{\Longleftrightarrow}$ ∂_b and orientation of page give orientation of M).
$\xi = \ker\alpha$ is supported by (B,p) if
- α induces positive orientation of M.
- $d\alpha|_{page}$ is symplectic and positive.
- $\alpha|_B$ is contact and positive. (this is harder when $n > 3$)

Theorem: M^n, n odd, closed, M has OBD $(B(\Sigma,\phi), p(\Sigma,\phi))$ with

- Σ compact, has exact symplectic form $\omega = d\beta$;
- Liouville vector field for ω given by $\iota_Y \omega = \beta$ is transverse to $\partial\Sigma$, pointing out.
- ϕ is a symplectomorphism.

Then, M admits ξ supported by (B,p).

Theorem: [Giroux - Mohsen]: To any contact structure ξ on closed $M^{n\geq 3}$, there exists an OBD (B,p) of M supporting ξ.

Rem: can see this as boundary of Lefschetz fibration, with the interior being $\mathbb{D}^2 \times$ binding. Then we can use Seidel's machinery. (Laura)

Rem: there exists an open book decomposition on a n-dim manifold M if

- n is odd
- $n \equiv 2 \bmod 4$ and M is simply connected.
- $n \equiv 0 \bmod 4$ and the signature is 0.

Sacha Zamorzaev - From Stein to loose Legendrians

Outline:
- Stein manifolds, Weinstein manifolds
- Building Weinstein manifolds using handles attachment // Morse
- h-principles for Weinstein manifolds
- Loose Legendrians.

Definition: a Stein manifold is an affine complex manifold, ie a properly embedded complex submanifold of $\mathbb{C}^n$.
ex: smooth affine varieties, or T^*X for X compact.
NOT Stein: anything complex compact (cf max principle)

Q: When is a complex manifold V Stein?
A: Iff $\exists\, \phi : V \to \mathbb{R}$ st ϕ is exhausting and J convex (plurisubharmonic)

Definition: $\phi : V \to \mathbb{R}$ is exhausting if it's proper and bounded below, ie $\phi^{-1}((-\infty,x))$ are compact sets exhausting the manifold. $(d^{\mathbb{C}}\phi = d\phi \circ J)$

Definition: ϕ is J-convex if $\omega_\phi := dd^{\mathbb{C}}\phi$ gives a Riemannian metric $\omega_\phi(-,J-)$.
ex: for $V = \mathbb{C}^n$, take $\phi = |z|^2$. For anything embedded, take the restriction of this ϕ.

Rem: any Stein manifold is symplectic ($\omega_\phi = dd^{\mathbb{C}}\phi$ is symplectic).

Q: When is a symplectic manifold Stein?
A: Always* (* = except when it obviously can't be)

Theorem: if $\overset{2n}{\dim} V > 4$ with almost complex structure J and $\phi : V \to \mathbb{R}$ is an exhausting Morse function with no critical point of index $> n$, then we can deform (J, ϕ) to a Stein structure (ie a couple (J,ϕ) satisfying the above conditions).

why is it obvious we can't have index $> n$? Because stable mfds are isotropic

$\Rightarrow T^*X$, X compact, is Stein.

Definition: a Weinstein structure on V^{2n} is

- $*$ ω a symplectic form
- $*$ ϕ an exhausting generalized Morse function
- $*$ X a complete vector field on V which is
  - Liouville (wrt ω)
  - gradient-like (wrt ϕ).

The reason we add this 3rd condition is to be able to recover the $d^c\phi = \beta$ $d\beta = \omega$

A generalized Morse function is one that admits cubic critical points, or equivalently whose critical points are either nondegenerate or birth-death.

Definition: X is Liouville means that $d\lambda = \omega$, for $\lambda := \omega(X, -)$.
This λ will be related to $d^c\phi$ of the Stein case.

Definition: X is gradient-like for ϕ means that

- $*$ ϕ increases along X
- $*$ $X = \nabla\phi$ near the critical points of ϕ.

We have maps

$$\begin{tikzcd}
 & \text{Stein structures} \arrow[d, "W"] & (V, J, \phi) \arrow[d, mapsto] \\
(\text{Liouville manifolds}) & \text{Weinstein structures} \arrow[l] \arrow[d] & (V, \underset{\omega}{-dd^c\phi}, \underset{\lambda}{-d^c\phi}, \underset{\phi}{\phi}) \arrow[d, mapsto] \\
 & \text{Gen. Morse functions} & (V, \phi)
\end{tikzcd}$$

We don't keep track of the J in Weinstein structures because it's recoverable up to contractible choice (of metric).

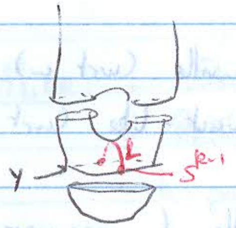
Theorem: W is an isomorphism on π_0.

Conjecture: W is a weak homotopy equivalence. The proof for the parametric case fails because we move critical points around...

Restricting, we have a similar statement about flexible Weinstein structures, and Morse (gen.) functions of index $\leq n$.

Perturbing Weinstein manifolds:

- Perturb ϕ so that no critical points are on the same level



We understand this via only the map $S^{k-1} \hookrightarrow (Y, \lambda)$.
L is isotropic (consider flow of X, converge to crit point. The only way for this not to blow up is that it is isotropic)

If $k<n$, we have an h-principle ($k<n$ comes from holonomic approximation)

$$\left\{\begin{tikzcd} TS^{k-1} \arrow[r, "\text{isotropic}", "df"'] \arrow[d] & TY \arrow[d] \\ S^{k-1} \arrow[r, hook, "f"] & Y \end{tikzcd}\right\} \longleftrightarrow \left\{\begin{tikzcd} TS^{k-1} \arrow[r, "\text{isotropic}", "F"'] \arrow[d] & TY \arrow[d] \\ S^{k-1} \arrow[r, hook, "f"] & Y \end{tikzcd}\right\}$$

It also holds if $k=n$ and f is loose.

Loose roughly mean that we can find a small disk in S^{k-1} where it looks like  in $\mathbb{R}^n$, times I^{n-2}, in $\mathbb{R}^{2n-1} \subseteq Y$. Equivalently, there should be a . The  should be thick enough: the length ℓ of the interval we cross with should have $\ell^2 > 2a$, where a is the action of the Reeb chord: . We can always find this picture $\times I$, but not $\times I$ for a long enough (ie $\ell^2 > 2a$) interval.

Oleg Lazarev – Symplectic cohomology and wrapped Floer cohomology

Outline: I. Floer cohomology for closed manifolds
II. Symplectic cohomology
III. Properties of SH
IV. Computations
V. Wrapped Floer cohomology

Ref: Seidel's, Wendl's and Oancea's surveys.

I. Floer cohomology for closed manifolds:

- (M^{2n}, ω) closed symplectic manifold, $H : S^1 \times M \to \mathbb{R}$ gives X_H the Hamiltonian vector field st $\omega(X_H, \cdot) = dH$.
- Original motivation: Arnold conjecture
  $\#(\text{time-1 orbits of } X_H) \geq \min\limits_{f \text{ Morse on } M} \#(\text{crit pts of } f)$.

Chain complex:

- CF generated by contractible time-1 orbits (as a $\mathbb{K}$-vector space)
- Grading: if $\langle c_1(M), \pi_2(M)\rangle = 0$, then $\mathbb{Z}$-grading

$D =$  orbit $\rightsquigarrow$ paths of lin symplecto of $(\mathbb{R}^{2n}, \omega_{std}) \rightsquigarrow \mu_{CZ}(\gamma; D) \in \mathbb{Z}$.

If other  $S^2 = \tilde{D}$, then $\mu_{CZ}(\gamma, \tilde{D}) = \mu_{CZ}(\gamma, D) + 2\overset{0}{\overset{\parallel}{\langle c_1(M), S^2\rangle}}$.

Differential:

- Auxiliary data: J_t, $t \in S^1$, compatible with ω
  $\rightsquigarrow \mathcal{M}(\gamma_-, \gamma_+) = \{ u : \mathbb{R}_s \times S^1_t \to M, \frac{\partial u}{\partial s} + J_t(u(s,t))\frac{\partial u}{\partial t} - X_H(t, u(s,t)) = 0$
  and $\lim\limits_{s \to \pm\infty} u(s, \cdot) = \gamma_\pm \}$.
- $\mathcal{M}(\gamma_-, \gamma_+)/\mathbb{R}$ manifold of dimension $|\gamma_-| - |\gamma_+| - 1$. We'd like to know when this is compact.
- Assume ω symplectic aspherical $\langle \omega, \pi_2(M)\rangle = 0$
  $\rightsquigarrow A_H : \Lambda^{cont} M = \{\text{contr. loops in } M\} \to \mathbb{R}$
  $\gamma \longmapsto -\int_{D^2} \bar{\gamma}^* \omega + \int_{S^1} H(t, \gamma(t))\, dt$

$\rightsquigarrow u \in \mathcal{M}(\gamma_-,\gamma_+)/\mathbb{R}$, $E(u) := \int \left|\frac{\partial u}{\partial s}\right|^2 ds dt = A_H(\gamma_-) - A_H(\gamma_+)$.

We have an a priori energy bound, given a codim-1 compactification of $\mathcal{M}(\gamma_-,\gamma_+)/\mathbb{R}$:

1) Bubbling : 

(doesn't happen if aspherical)

2) Floer breaking 

The bubbling has codim 2, so does not entrave $d^2=0$ (but the higher μ^k's). This is if the manifold is closed. If open, we have disk bubbling, which is codim 1 and can create $d^2 \neq 0$.

- If $|\gamma_-| - |\gamma_+| - 1 = 0$, then $\mathcal{M}(\gamma_-,\gamma_+)/\mathbb{R}$ is already compact, and we can define $d\gamma_+ = \sum_{\substack{\gamma_- \text{ s.t.} \\ |\gamma_-|-|\gamma_+|-1=0}} \#\left(\mathcal{M}(\gamma_-,\gamma_+)/\mathbb{R}\right)\gamma_-$.
- $d^2=0$ 

(broken Floer trajectory = boundary of 1-dim moduli space)

$\Rightarrow$ Define $HF(M,H,J_t) = H(CF,d)$.

Invariance:

- Independant of H = Hamiltonian function : count parametrized Floer trajectories for $H_{s,t}$; get continuation maps

$$\begin{tikzcd} HF(H_0) \arrow[r] \arrow[rr, bend right, "id"'] & HF(H_1) \arrow[r] & HF(H_0) \end{tikzcd}$$

Theorem: $HF^\bullet(M;\mathbb{K}) \cong H^{\bullet+n}(M;\mathbb{K})$.

This implies a weak version of the Arnold conjecture.

Proof: $HF(M;\mathbb{K})$ is independant of H. Choose a C^2-small, time-independant H.
↳ so we don't get time-1 non cst orbits

- Ham orbits = critical points of H
- Floer trajectories = gradient flow of H

$\Rightarrow HF(M;\mathbb{K}) \cong H^{\bullet+n}_{Morse}(M) \cong H^{\bullet+n}(M)$. □

II. Symplectic cohomology.

Definition: Liouville domain (M^{2n}, λ) ← Liouville 1-form, with non-empty boundary, with

1) $\omega = d\lambda$
2) X_λ = Liouville vector field defined by $d\lambda(X_\lambda, \cdot) = \lambda$ is transverse to $\partial M = Y$.

$\Rightarrow (Y^{2n-1}, \lambda|_Y)$ is contact. (sometimes, we'll write $\alpha = \lambda|_Y$)

Rem: $\mathcal{L}_{X_\lambda} \lambda = \lambda$, $\mathcal{L}_{X_\lambda} \omega = \omega$.
$\rightsquigarrow$ Neighbourhood of Y is $((-\varepsilon, 0] \times Y, d(e^t \lambda))$
$\rightsquigarrow$ Attach symplectization $([0,\infty) \times Y, d(e^t \lambda|_Y))$, called $\hat{M} = M \cup ([0,\infty) \times Y)$

Rem: actually we can flow X_λ to $-\infty$, so we actually have the entire symplectization.

ex: • $\mathbb{C}^n$, $\lambda = \frac{1}{2}(\sum x_i dy_i - \sum y_i dx_i)$ $\quad X_\lambda = \sum x_i \partial_{x_i} + \sum y_i \partial_{y_i}$
• T^*Q, $\lambda = \sum p_i dx_i$ $\quad X_\lambda = \sum p_i \partial_{p_i}$

Problems: 1) Floer trajectories between two fixed hamiltonians orbits might not be contained in a compact space
$\rightsquigarrow$ compactness fails and $d^2 \neq 0$.
2) Parametrized Floer trajectories have the same problem.

Assume $H : \hat{M} \to \mathbb{R}$ time-independant (in the end, it will have to be time-dependant such that $H|_{[0,\infty) \times Y}$ is of the form $H(r,x) = h(e^r)$. Then, $X_H = h'(e^r) R_\lambda$ where R_λ is the Reeb vector field of the contact form $\lambda|_Y$.
If $y(t) \subseteq Y$ is a T-periodic Reeb orbit, then $(r, y(T.t))$ is 1-periodic, where r is chosen so that $h'(e^r) = T$.
If H is C^2-small in M (not $\hat{M}$), then the Hamiltonian orbits are Morse crit. point

Solution to 1): take J of contact type, ie $J\partial_t = R_\alpha$ and $J_{|\ker\alpha}$ is compatible with $d\alpha$. $\rightsquigarrow$ max principle, so all Floer trajectories for given Hamiltonian orbits stay in compact region. So, $d^2 = 0$.

?

Solution to 2): if H_s has $\frac{\partial H}{\partial s} \geq 0$, then a max principle also holds.
$\rightsquigarrow$ get $HF(H_0) \to HF(H_1)$.

Usually, people look at $HF(H_k)$, where H_k means that the hamiltonian has constant slope at ∞ : $h'(e^r) = k$ at ∞. As we increase k, we have continuation maps as in the previous paragraph.

Definition: $SH(\hat{M}) := \varinjlim_{k} HF(H_k)$.

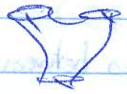
Intuition: $HF(H_k)$ only sees Reeb orbits with period $\leq k$. The alternative is to take a quadratic hamiltonian.

Rem: this is invariant under exact symplectomorphisms of $\hat{M}$.

III. Properties:

Prop 1: $\phi : H^{\bullet+n}(M;\mathbb{K}) \longrightarrow SH(M)$ ~~inclusion~~.

Rem: if not isomorphism, then for any contact form on $(Y, \ker \lambda|_Y)$ has a Reeb orbit.

Prop 2: SH has a product , and even a TQFT.

We need the marking to get a parametrization, because all things come parametrized.  gives a unit in $SH^{-n}(M)$, image of the unit in $H^0(M) \to SH^{-n}(M)$.

Prop 3: If we rotate the marking , we get map $\Delta : SH^{\bullet}(M) \to SH^{\bullet-1}(M)$.

$(SH, \cdot, \Delta)$ is a BV-algebra.

$\rightsquigarrow [a,b] = \Delta(a \cdot b) - \Delta(a) \cdot b - a\,\Delta(b)$ is a Lie bracket.

Prop 4: (Viterbo functoriality) let $N \subseteq M$ be a Liouville subdomain. Then, we get a unital ring map $SH(M) \to SH(N)$.

Corollary: $N \subseteq M$, then $SH(M) = 0 \Rightarrow SH(N) = 0$.

Proof: $\underset{\text{since } SH(M)=0}{0 = 1_M} \longmapsto 1_N = 0$, so $SH(N) = 0$. □

IV. Computations:

Proposition: M = spin, closed

free loop space

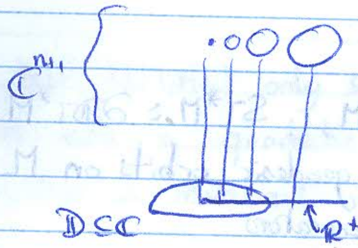
$$\begin{tikzcd} SH^\bullet(T^*M) \arrow[r, phantom, "\cong"] & H_{-\bullet}(\Lambda M; \mathbb{K}) \\ H^{n+\bullet}(T^*M) \arrow[u] \arrow[r, phantom, "\cong"] & H^{n+\bullet}(M) \arrow[u, "\text{ind. of cst map}"'] \end{tikzcd}$$

Proof: pick g a metric on $M \rightsquigarrow \mathbb{D}T^*M$, $ST^*M = \partial\mathbb{D}T^*M$. The Reeb orbits on (ST^*M, α_g) are geodesic orbits on M. So we see that they have the same generators. □

Corollary: if $SH(N)=0$, then N has no closed exact Lagrangian.
Proof: $L \subset N \rightsquigarrow T^*L \subset N$ is a Liouville subdomain. If $SH(N)=0$, then by previous corollary we should have $SH(T^*L)=0$, but it's not by the previous proposition. ↯ □

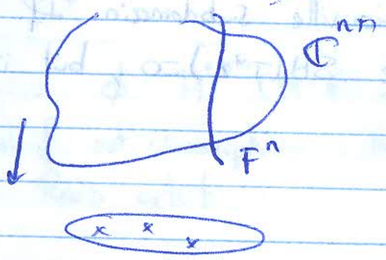
Daniel Álvarez-Gavela - The A_m^n - Milnor fibre

Consider the map $\mathbb{C}^{n+1} \to \mathbb{C} : (z_1, \dots, z_{n+1}) \mapsto z_1^2 + \dots + z_{n+1}^2$.
If we look at real solutions, we get spheres $x_1^2 + \dots + x_{n+1}^2 = r$.
above $\mathbb{R}_+$



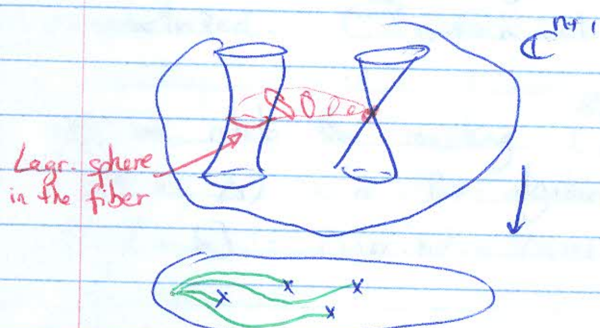
The collection of all these spheres is a $n+1$ Lagrangian ball.
This will be our local model.

In general, polynomials $\mathbb{C}^{n+1} \to \mathbb{C}$ give Lefschetz fibrations.



Take $p(z_1, \dots, z_{n+1}) = z_1^{m+1} + z_2^2 + \dots + z_{n+1}^2$.
There is a multiplicity m crit point, so we change p to separate these m crit points:
$\tilde{p} = p_1(z_1) + z_2^2 + \dots + z_{n+1}^2$.
↑ generic degree $m+1$.

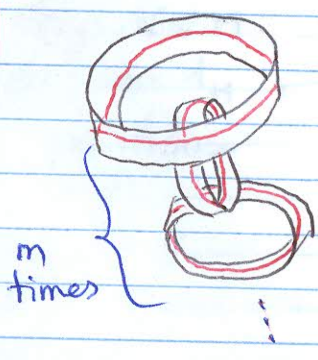
Play the same game as before, with critical points



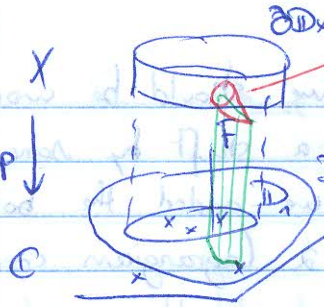
But, what is this fiber? We have m critical values, so we get m spheres. How do they intersect? Just like in the Dynkin diagram A_m: •—•— ⋯ —•—•

$$\# S_i \cap S_j = \begin{cases} 1 & \text{if } |i-j|=1 \\ 0 & \text{if } |i-j|>1 \end{cases}$$

Actually, we get a plumbing of m T^*S^n:



Take like DT^*S^n and glue them together by gluing base to fiber.



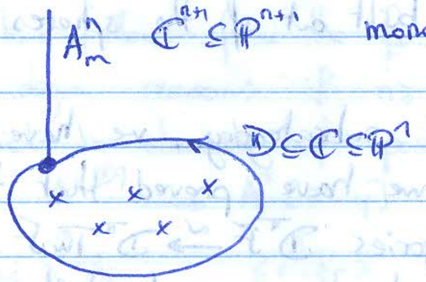
Legendrian sphere in the contact manifold $\partial(p^{-1}(\mathbb{D}))$

$$\partial(p^{-1}(\mathbb{D})) = \mathbb{D}\times\partial F \cup \partial\mathbb{D}\times F$$

As we go around a singularity, the sphere in the fiber glues back to itself by a Dehn twist. The monodromy around a critical point is a Dehn twist about the Lagrangian sphere

If we increase $\mathbb{D}_1$ to $\mathbb{D}_2$, what happens? Take a green path from the critical point to $\partial\mathbb{D}_1$. We see on the picture that we attach a handle.

Back to the A^n_m-Milnor fibre $p: \mathbb{C}^{n+1}\to\mathbb{C}$. What is the monodromy around this loop?



We can homotope this loop to a constant one in $\mathbb{P}^1$, so the monodromy $A^n_m \to A^n_m$ must be the identity. (*)

Fukaya category of A^n_m:

$$\begin{tikzcd}
L \arrow[rr] & & \tau_S(L) \arrow[dl] \\
& \mathrm{Hom}(S,L)\otimes S \arrow[ul] &
\end{tikzcd}$$

is an exact triangle.

direct sum of copies of S

We can compose things:

$$\begin{tikzcd}
L \arrow[rr] & & \tau_S(L) \arrow[rr] \arrow[dl, "{[1]}"'] & & \tau_{S'}(\tau_S(L)) \arrow[dl, "{[1]}"'] \\
& \mathrm{Hom}(S,L)\otimes S \arrow[ul] & & \mathrm{Hom}(S',\tau_S(L))\otimes S' \arrow[ul] \arrow[dl] & \\
& & {[\dots]} \arrow[ul] & &
\end{tikzcd}$$

built out of S and S'

Use triangulated categories axioms.

If we compose with all the spheres:

$$\begin{tikzcd}
L \arrow[rr] & & \tau_{S_1}(\dots(\tau_{S_m}(L))\dots) \overset{\text{by } (*)}{=} L \arrow[dl] \\
& \text{built out of } S_1,\dots,S_m \arrow[ul] &
\end{tikzcd}$$

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The top map is the identity, but we should be working with graded Lagrangians, so it's actually a shift by some $N \neq 0$. So far we're working with Tw Fuk, since we added the bottom terms in the Seidel triangles. Geometrically, cone of 2 Lagrangians correspond to the symplectic connected sum .

Do it several times: $L \xrightarrow{[kN]} L$. We have $HF^*(L,L) \simeq H^*(L)$. So for large enough k, this map has to be 0 on the level of homology, for degree reasons. So:

$$\begin{tikzcd} L \arrow[r, "0"] & L \arrow[dl] \\ \text{built out of } S_1, \dots, S_m \arrow[u] & \end{tikzcd}$$

$\Rightarrow L$ is a direct summand of that thing built out of the spheres.

So if $\mathcal{F} \subseteq$ Fuk is the full subcategory, we have $\mathrm{Tw}\,\mathcal{F} \subseteq \mathrm{Tw}\,\mathrm{Fuk}$. If we pass to split-things, we have proved that this inclusion induces an equivalence of categories $D^\pi \mathcal{F} \xrightarrow{\sim} D^\pi \mathrm{Tw}\,\mathcal{F}$.
$D^\pi \mathcal{F}$ has a finite number of objects, so is tractable.

Consider $F = \bigoplus_{i,j} \mathrm{Hom}(S_i, S_j)$ the A_∞-algebra built out of the A_∞-category $\mathcal{F}$. We pass to the cohomology $H^\circ(\mathcal{F})$. We know that $HF^\circ(S_i, S_j) \simeq H^{\circ + *}(S_i)$ has 1 generator in degree 0 and 1 generator in degree n.
We also know that $\underset{i \neq j}{HF^\circ(S_i, S_j)} = \begin{cases} \dim 1 & |i-j| = 1 \\ \dim 0 & |i-j| > 1 \end{cases}$

This gives us a quiver.

Choose the degrees of the vertices so that the top arrows have degree 0. By PD, bottom ones have degree n: $HF^k(S,L) \otimes HF^{n-k}(L,S) \simeq H^n(L,L)$.
Reshift everything (assume degree is even): 

So $H^\circ(F) =$ the Path algebra of the A_m quiver $\Big/ \begin{matrix} (i,i+1,i+2) = (i,i-1,i-2) = 0 \\ (i,i+1,i) = (i,i-1,i) \end{matrix}$.

So we have
$$\begin{aligned} &\mathcal{F} \hookrightarrow \mathcal{F}uk \\ &Tw\,\mathcal{F} \hookrightarrow Tw\,\mathcal{F}uk \\ &D^\pi \mathcal{F} \xrightarrow{\sim} D^\pi \mathcal{F} \end{aligned}$$

We would like to know how much this determines the Fukaya category.

$A = H^\circ(\mathcal{F})$ is an associative graded algebra.
<u>Claim</u>: it is intrinsically formal for $n > 1$: any A_∞-algebra $\mathcal{A}$ such that $H^\circ(\mathcal{A}) \simeq A$ (as associative graded algebras) is quasi-isomorphic to A (as A_∞-algebras).

This follows from the fact that $HH^q(A, A[2-q]) = 0$. This is just for degree reasons if $n \geq 4$. That's because (theorem) we can restrict to elements of positive degree:

$$\underset{\deg \geq \frac{nq}{2}}{\underset{\uparrow}{A^{+\otimes q}}} \longrightarrow \underset{\leq n+q-2}{\underset{\uparrow}{A^{+}[2-q]}}.$$

Kyler Siegel - Flexibility and holomorphic curve invariants

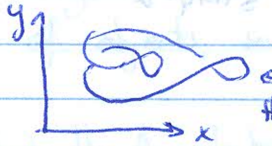
Metaprinciple #1: Let O be a symplectic object. If O is flexible, we expect hol. curve invariant (O) to be trivial.

Metaprinciple #2: all hol. curve invariants$(O) = 0 \Rightarrow O$ is flexible.

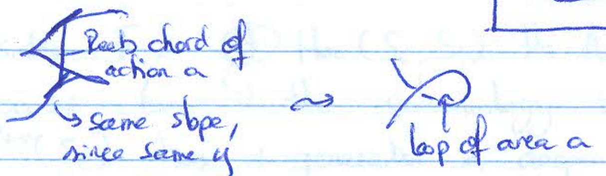
Q: How to tell two flexible things apart?
A: Easy! They're supposed to be the same $\Leftrightarrow$ they are formally the same.

1. **ex:** LCH(loose legendrian) = 0 (ie, 0-dim vector space)
Suppose we have a Legendrian knot $\Lambda \subseteq \mathbb{R}^3_{std}$:

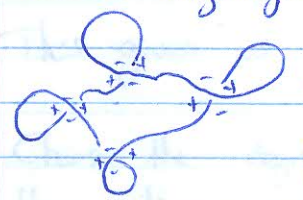
Fact: Λ has a zigzag $\Rightarrow LCH(\Lambda) = 0$.
In the Lagrangian projection:



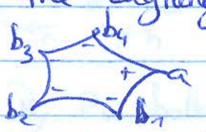
← zigzag becomes this in this projection.



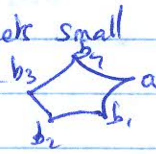
$LCH(\Lambda) =$ free associative $\mathbb{K}$-algebra generated by Reeb chords, ie intersection points in the Lagrangian projection.



Give signs to crossings, by some right-hand rule.
The differential counts polygons in the Lagrangian projection with only one + sign:

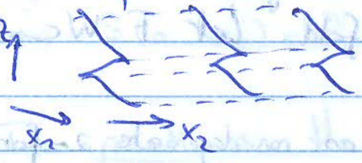


$da = b_1 b_2 b_3 b_4 + \ldots$

By shrinking the zigzag , we can make the loop  in the Lagrangian projection as small as we want. Call z this $\cap$ point.
So, for the polygon , we have $dz = 1 + \ldots$
By shrinking it so that the action gets small (by Stokes, the action is the area of ), and in general for  $A(a) - \sum A(b_i)$ is some energy, hence ≥ 0. But we can make $A(a)$ as small as we want. So there can't be anything else, in

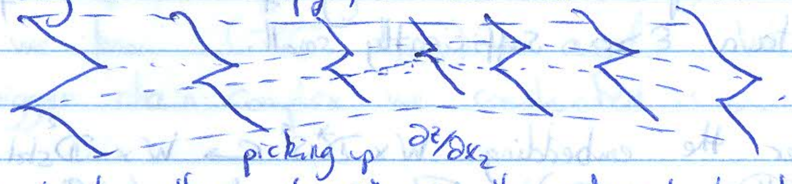
$dz = 1 + \dots$ This shows that 1 is nullhomologous in this algebra, so the unit is 0. So, LCH has to be 0.

In higher dimensions, a zigzag is a "product" of a zigzag and a bunch of intervals.



(front projection in $\mathbb{R}^5$)
This is a "loose chart".

By a Legendrian isotopy, we can make the center small:



We can "make things trivial" in the interval direction, so that the count of holomorphic disks reduces to the count of holomorphic disks in dimension 3.

2. SH(Flexible Weinstein domain) = 0:

Actually, SH(subcritical W. d.) = 0.

Theorem (Cieliebak): for W^{2n} subcrit W. d., $W \underset{\uparrow}{\cong} M^{2n-2} \times \mathbb{D}^2_{std}$.
deformation equivalent

Theorem (Oancea): over a field, $SH(W_1 \times W_2) \cong SH(W_1) \otimes SH(W_2)$.

Fact: $SH(\mathbb{D}^2_{std}) = 0$. True for higher dim disks true. But there exists high dim non standard discs with $SH \neq 0$.

Rough sketch: $\mathbb{D}^2_{std}$, $H_\tau = \tau |z|^2$ (τ generic positive real number)

The Hamiltonian flow of H_τ is a rotation.

$$SH(\mathbb{D}^2_{std}) = \lim_{\tau \to \infty} \underbrace{HF(H_\tau)}_{\text{1-dim, gen by cst orbit at origin}}$$

Q: what is the index of the constant orbit of H_τ?

A: it actually grows with τ.

Proof #1:

Theorem [Eliashberg - Murphy]. "flexible W.d.'s are easy to exact symplectically embed $\left\{\begin{matrix}\text{somewhere}\\ \text{anywhere}\end{matrix}\right.$". This was the slogan. Here is the theorem:

Let (W^{2n}, λ, ϕ) be a flexible W.d. (hence compact by def) and (X^{2n}, θ) an exact symplectic manifold. Let $F: W \hookrightarrow X$ be a smooth embedding, and assume $F^* d\theta \simeq d\lambda$

isotopic as non degenerate 2-forms.

Then, F is isotopic to an embedding $f: W \hookrightarrow X$ st $f^*\theta = \varepsilon\lambda + d(fct)$, for a certain $\varepsilon > 0$ sufficiently small.

Ok. Consider the embedding $W \times D^*S^1 \hookrightarrow W \times \mathbb{D}^2_{std}$ which is id × ( $\hookrightarrow$ ). This can be made a symplectic embedding. But $D^*S^1 \hookrightarrow \mathbb{D}^2$ is definitely NOT an exact symplectic embedding: the 0-section in D^*S^1 is an exact lagrangian C; hence $\int_C pdq = 0$. But the image of this loop in  bounds a disk with positive area, so by Stokes we have $\int_{f(C)} pdq \neq 0$. So, not exact.

By Eliashberg and Murphy, we get an exact symplectic embedding $(W \times D^*S^1, \varepsilon(\lambda + pdq)) \hookrightarrow W \times \mathbb{D}^2_{std}$

Since $SH(W \times \mathbb{D}^2_{std}) = 0$ (cf subcritical), and because of the Viterbo transfer map (which is a unital ring map) $SH(W \times \mathbb{D}^2_{std}) \to SH(W \times D^*S^1)$, hence we must have $SH(W \times D^2S^1)$. But this is $SH(W) \otimes SH(D^*S^1)$. But $SH(D^*S^1)$ is the homology of the loop space, so we must have $SH(W) = 0$.

□

Proof #2: W flexible W.d. Assume the simplest case $W = \left(\overset{\overset{M}{\shortparallel}}{\begin{matrix}\text{subcritical}\\ \text{W.d.}\end{matrix}}\right) \cup \left(\begin{matrix}\text{flexible}\\ \text{handle}\end{matrix}\right)$, attached along a legendrian sphere $\Lambda \subseteq \partial M$.

Λ is loose (this is the def of flexible), so $LCH(\Lambda) = 0$. Call $\mathcal{G}$ the vector space generated by the Reeb chords of Λ.

$LCH(\Lambda) = H(\overset{\text{tensor}}{T}\mathcal{G}, d_{LCH})$.

Let $\Omega\mathcal{G} = \mathcal{G}[1] \otimes T\mathcal{G}$.

Consider the following chain complex: $\left(TG \oplus \Omega, \begin{pmatrix} d_{LCH} & P \\ & d_{cyc} \end{pmatrix}\right)$.
Write a typical element of ΩC as $\hat{a}w$ for $a \in G[1]$ (the hat is just a notation to remember that it comes from $G[1]$).

$P : \Omega G \longrightarrow TG : \hat{a}w \mapsto aw \pm wa$ (Some Koszul signs)

$d_{cyc} : \Omega G \longrightarrow \Omega G : \hat{a}w \mapsto \pm \hat{a}\, d_{LCH}(w) \pm S(d_{LCH}(a))w$

where $S(a_1 \dots a_k) = \hat{a}_1 a_2 \dots a_k + \hat{a}_2 a_3 \dots a_k a_1 + \dots + \hat{a}_k a_1 \dots a_{k-1}$.

The point is that $SH(W) = H$(this chain complex), by the handle attachment of the beginning. (this uses the P)

Goal: we know (TG, d_{LCH}) is acyclic. We want to show that this bigger chain complex we constructed is 0.

(TG, d_{LCH}) is a subcomplex of the bigger one (because the differential is upper triangular), so quotient by it. So, it suffices to show that $(\Omega G, d_{cyc})$ is acyclic.

Claim: there is some increasing filtration on G $F_pG \subseteq F_{p+1}G \subseteq \dots$ such that d_{LCH} is strictly decreasing wrt this filtration. It is just the action filtration.

Consider the filtration ~~$F_p(TG \oplus \Omega G) = F_p TG \oplus \Omega G$~~

$F_p(G[1] \oplus TG) = F_pG \oplus TG$.

What are the associated graded parts?

$$\underbrace{\frac{F_p(G)}{F_{p-1}(G)} \oplus TG}_{\text{acyclic with induced differential, essentially because } TG.}$$

Sheel Ganatra, Mark McLean - Overview of the topic:

First we'll describe the set of objects we want to study; then the tools for it.

Stein manifold = closed properly embedded complex submanifold of $\mathbb{C}^N$
ex: affine varieties, $\{\sin y = x^2\} \subseteq \mathbb{C}^2$.
These have a symplectic form: for $X \subseteq \mathbb{C}^N$, $\omega_X = \omega_{std}|_X$. We study these up to symplectomorphism, or some kind of deformation equivalence.
Other way of describing it: Stein manifold = complex manifold with an exhausting plurisubharmonic function $\rightsquigarrow \omega_X = -dd^{\mathbb{C}}\phi$
(↳ proper, bounded below   (↳ $-dd^{\mathbb{C}}\phi(v, Jv) > 0$, where $d^{\mathbb{C}}\phi(v) = d\phi \circ J(v)$)
Think about (X, ω_X) in terms of a handle decomposition coming from ϕ (assume ϕ is Morse, generically). Let $X^{\leq c} = \phi^{-1}(-\infty, c]$; it is a compact manifold which is symplectic, with contact boundary. This is described in terms of a "Weinstein handle decomposition".

Lefschetz fibration: "Weinstein handle decomposition with extra structure"; it comes with a map $\pi : X \to \mathbb{C}$ which is a fibration with some "generic" isolated singularities, and the smooth fibers are symplectic. The handles are described in terms of Lagrangian spheres inside a fixed smooth fiber. These fibers should be thought of as Stein manifolds (1 dimension lower). So this is good for describing high-dim Stein manifolds in terms of lower-dim ones.

ex:

Flexible phenomena: all symplectic aspects are described purely topologically.
ex: flexible Stein manifolds.
ex: if X is any Stein manifold and $X \times \mathbb{C}$ is diffeomorphic to $\mathbb{C}^N$, then $X \times \mathbb{C}$ is symplectomorphic to $\mathbb{C}^N$.

Rigidity: ex: exotic Stein manifolds, and lots of purely symplectic phenomena

Geometry table:

	(Wein)Stein $\xrightarrow{\partial}$	Contact
Presentation as	Handlebody ↳ subcritical (topology) ↳ critical ⇝ Legendrians / Lagrangians	Surgery diagram
or	Lefschetz fibration ↳ Lagrangians ↳ Vanishing cycles / Thimbles (Lagrangians)	Open book

Flexibility [Cieliebak - Eliashberg, Murphy]: there is a way of detecting if X is flexible using the presentations.

Rigidity phenomena: symplectic invariants coming from J-hol curves give ways of detecting "symplectic" data

ex: (X, ω) (Wein)Stein ⇝ $SH^*(X)$ [Viterbo, (Cieliebak)-Floer-Hofer]

symplectic cohomology (also called $SH_*(X)$, but same group! Different grading].

Theorem: $SH^*(X) \neq 0$ ⇝ X not flexible.

Rem: $SH^*(X \times \mathbb{C}) = 0$

There is a map $H^*(X) \to SH^*(X)$, but it is almost never an isomorphism.

Chain complex: $C_{\text{Morse}}(X) \oplus$ "Reeb orbits on ∂X" ⇝ ∞ rank!

More generally: X Weinstein manifold

	"relative", "open string"	"closed string"
	$L_i \subseteq X$ potentially non compact lagr	unital algebra ($SH = 0 \iff 1 =$
"Hamiltonian"	↳ $HW^*(L_1, L_2)$ wrapped lag Floer hom.	$SH^*(X)$ + S^1-equiv theories
X completed: convex symplectic manifold	↳ $\mathcal{W}(X)$ wrapped Fukaya category	
SFT*	$\Lambda \subseteq Y$ Legendrian	$SH^*(X)$ (SFT version)
Y contact manifold, ⋮ X symplectic filling + "virtual techniques"	↳ $LCH(\Lambda)$ Leg. contact homology + sometimes linearized/augmented by X with $\partial X = Y$ or L with $\partial L = \Lambda$	Contact homology of $Y := \partial X$ augmented/linearized

unital algebra + unital SH-module ⇒ $HW^* \neq 0 \Rightarrow SH^* \neq 0$

multiply

"≅" "open-closed isomorphisms": conjecture

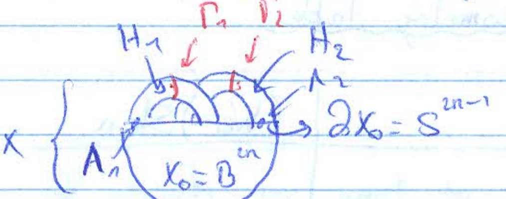
"="

mult.

o-c isom: conjecture*

How do our presentations of manifold affect the story?
"Presentations of $X \rightsquigarrow$ simplified formulae for open-closed string invariants of X"

Handlebody / surgery picture:



H_1, H_2 handles determined by Λ_1, Λ_2 Legendrians
$LCH(\{\Lambda_i\})$ in ∂X_0 linearized by X_0

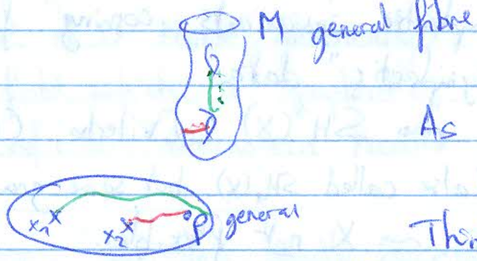
1) Theorem [Bourgeois - Ekholm - Eliashberg]: this gives us $SH^*(X)$ ← after surgery

2) The cocores Γ_1, Γ_2 that appear as new Legendrians after surgery ~~ka~~ get information from $LCH(\{\Lambda_i\})$: $HW^*(\Gamma_i, \Gamma_j)$.

We get a map 2) $\rightsquigarrow$ 1) by open-closed isomorphism.
and 1) $\longleftrightarrow$ 2) or rather, SFT analogues

Lefschetz fibration picture:

$X \xrightarrow{\pi} \mathbb{C}$



"Vanishing cycles" $\{V_i\}$

As $p \to \begin{cases} x_1 \\ x_2 \end{cases}$, the $\begin{cases} \text{green} \\ \text{red} \end{cases}$ Lagr. sphere collapses

Thimbles = traces of this collapsing. $\{\Delta_i\}$

So the data of $(M, \{V_i\})$ determines (X, π). Thimbles are a special case of cocores.
[Seidel, Abouzaid-Seidel]: Lagrangian Floer theory in $(M, \{V_i\})$ determine wrapped theory of $\{\Delta_i\}$.
Homology level: Maydanskiy (- Seidel).

Conjecture [Seidel, Abouzaid-Seidel, Ganatra, Abouzaid-Ganatra] $\mathcal{W}(X)$ determines $SH^*(X)$ when π exists via $OC: HH_*(\mathcal{W}) \xrightarrow{\sim} SH(X)$.
But there is a simplified way of computing this using

SH^* Seidel: $A \subseteq B$

$\{ \mathcal{F}(\pi) \longrightarrow \mathcal{F}(M) \}$

$\uparrow \Delta_i \longmapsto V_i$

Fukaya category of thimbles: finite rank Lag Floer homology groups

Plus, many examples!

[Maydanski - Seidel's exotic T^*S^n's)

Day 1): basic geometry (presentations) + SFT techniques

2) Rigidity: SFT, SH, HF^* ex: Milnor fibres A_n

3) Flexibility, Hw^*, relating presentations

4) SH^*, formulae for it (surgery + Lefschetz), examples

5) Applications.

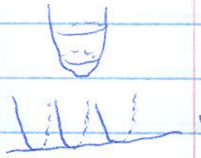
Sacha Zamorzaev: Weinstein structures

Definitions: a Liouville domain is a W compact symplectic with boundary $(W^{2n}, \omega, \partial W, \lambda, X)$ such that

- $d\lambda = \omega$
- $\omega(-, X) = \lambda$
- X outward pointing on ∂W

A Weinstein domain is a Liouville domain with a function $\phi : W \to \mathbb{R}$ which is Morse*, $\partial W = \phi^{-1}(c)$, and X is gradient-like for ϕ, meaning that ϕ is increasing along the flow of X: $d\phi(X)$ ~~>0~~ $\geq \delta(|X|^2 + |d\phi|^2)$.

A Weinstein manifold is built out of Weinstein domains:



ex: $\left(\mathbb{C}^N, \omega_{std} = \sum dx_i \wedge dy_i, \lambda_{std} = \frac{1}{2}\sum x_i dy_i - y_i dx_i, X = \frac{1}{2}\vec{r}, \phi = \frac{1}{4}|z|^2\right)$

non-ex: $(\mathbb{C}^N, \omega_{std}, \lambda_{st} = \sum x_i dy_i, X = \sum x_i \partial_{x_i}$

ex: Stein manifolds: proper complex submanifolds of $\mathbb{C}^N$, pullbacking everything from the 1st example.

ex: T^*M, M compact: $(T^*M, \omega_{std}, \lambda_{std}, X = \sum \xi_i \partial_{\xi_i}, \phi = \sum \xi_i^2 \overset{\wedge}{})$.
$+ f$ Morse on M.

T^*S^n is a cotangent bundle which is also an affine complex manifold: $\sum z_i^2 =$

$$\Leftrightarrow \begin{cases} \sum x_i^2 - y_i^2 = 1 \\ \sum x_i y_i = 0 \end{cases} \overset{\tilde{x}_i = \frac{x_i}{\sqrt{\sum y_i^2 + 1}}}{\Leftrightarrow} \begin{cases} \sum \tilde{x}_i^2 = 1 \\ \sum \tilde{x}_i y_i = 0 \end{cases} = T^*S^n.$$

As affine: $\phi = \sum |z_i|^2$, as cotangent bundle: $\phi = \sum |y_i|^2$

There is a canonical way to extend a Weinstein domain: let $\theta = \lambda|_{\partial W}$.

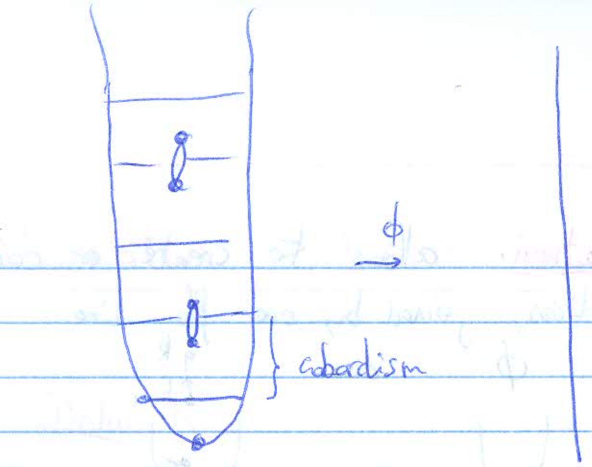
We have $0 \overset{\deg 2n+1}{=} (d\lambda)^{\wedge n} \wedge d\phi\,(X, \ldots) = \ldots \underbrace{d\lambda(X, -) \wedge d\lambda^{\wedge(n-1)} \wedge d\phi}_{= \lambda \wedge (d\lambda)^{\wedge(n-1)} \wedge d\phi} + \underbrace{(d\lambda)^{\wedge n}}_{\text{vol form}} \wedge \underbrace{d\phi(X)}_{>0}$

So $\lambda \wedge (d\lambda)^{\wedge(n-1)} \wedge d\phi$ is a volume form on W, so λ is a contact form on $\partial W =: Y$.

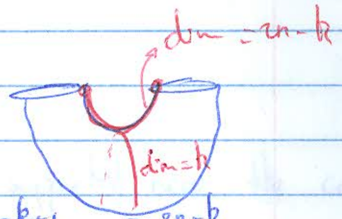
Attach on Y a $(Y \times \mathbb{R}_{>0}, \hat{\lambda} = e^t \theta, X = \partial_t, \phi = t)$.

ex: $\mathbb{C}^N$ is the completion of a ball of any radius around the origin.

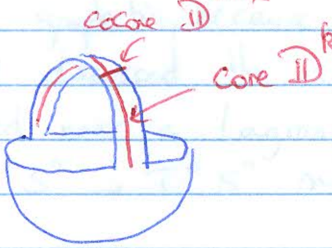
Morse theory:

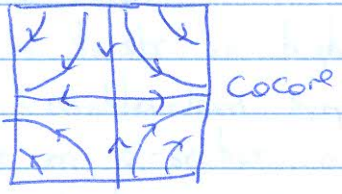


Look at ~~the~~ an ~~kth~~ index k critical point:



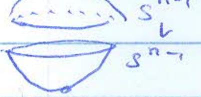
So we have a $\mathbb{D}^k \times \mathbb{D}^{2n-k}$ attached along $S^{k-1} \times \mathbb{D}^{2n-k}$





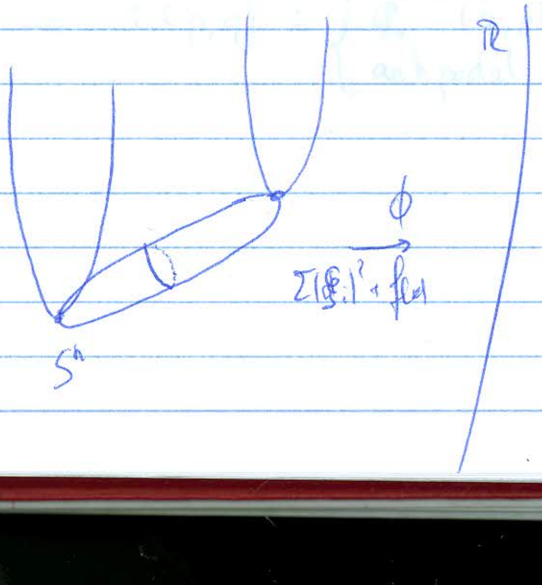
Fact: stable manifolds of Weinstein manifolds are isotropic.
Proof: $\mathcal{L}_X \omega = \omega$, so $\phi_t^* \omega = e^t \omega$. If we take a trajectory on the cocone, it gets to 0, so it had to be 0 at most. So the core is ~~Lagrangian~~ isotropic. □
$\hookrightarrow$ Core has dimension at most n.

Skeleton: "union of all stable manifolds".
And ∂core = attaching sphere is isotropic in the contact boundary.
If dim < n, topological attachment determines symplectic attachment:
So, Weinstein handles are flexible in dim < n.
In index n: $S^{n-1} \hookrightarrow Y$ is Legendrian.



Let's look at T^*S^n. First, S^n: crit pts of index 0 and n:

S^n = stable manifold of ~~[illegible]~~ 2nd crit pt

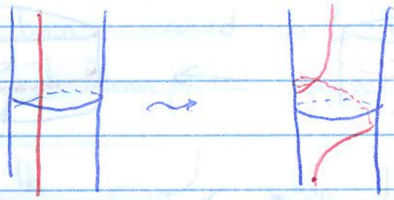


Morse cancellation: allows to create or cancel pairs of crit pts of adjacent indices, joined by one flow line.

Laura Starkston - Dehn twists.

1) What is a Dehn twist?

2-dim:



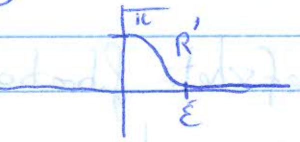
diffeo supported near the circle.

It's symplectic, because it preserves areas.
Arnold discussed the symplectic Dehn twists in higher dimensions: supported near a Lagrangian S^n (neighborhood symplectomorphic to T^*S^n)
$\varphi: T^*S^n \to T^*S^n$ such that * on 0-section: antipodal map
* outside compact set: identity.

We achieve it with a geodesic flow.
Normalized geodesic flow on $(T^*M, \Sigma\, dp_i \wedge dq_i)$. Consider $u(q,p) = \|p\| = \sqrt{\Sigma p_i^2} \leadsto du = \Sigma p_i dp_i / \sqrt{\Sigma p_i^2}$

$$\leadsto X_u = \frac{\Sigma p_i dq_i}{\sqrt{\Sigma p_i^2}}$$

; this is the (co)vector p seen in TM.
So, it gives the geodesic flow (not defined on the 0-section).
On S^n, the geodesics are great circles. Let φ_u be the geodesic flow. Choose a function



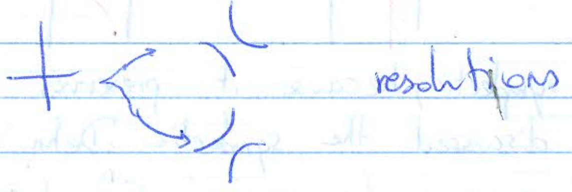
$$\leadsto \varphi^{t}_{R(u)}(q,p) \overset{\text{notations...}}{=} \varphi^{u}_{tR'(u)}(q,p).$$

$$\to \tau(p,q) = \begin{cases} \varphi_1^{R(u)}(q,p) & \text{elsewhere} \\ \text{antipodal map} & \text{on 0-section} \end{cases}$$

2) How do Dehn twists act on Lagrangians?

* If we Dehn twist about L_1 and L_2 disjoint, then L_2 is fixed.
* If L_2 intersects L_1 exactly once, they intersect transversely, we get the picture

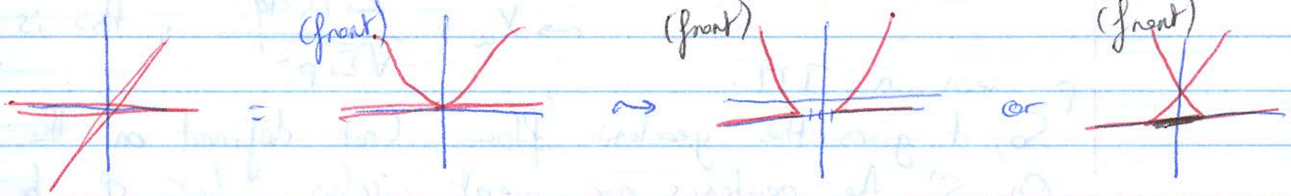
Polterovich surgery:



* If several intersections, connect them.

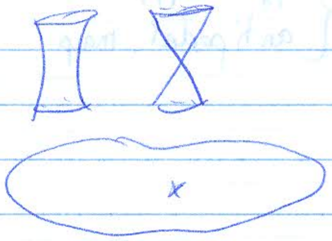
3) How do Dehn twists "act" on Legendrians?

Langrangian sphere $L \subseteq M^{\text{exact}}$ symplectic manifold : $\omega = d\lambda$.
Look at contactization $(M \times \mathbb{R}, \alpha = dz - \lambda)$



$$\begin{tikzcd}
& \text{Lefschetz fibrations} \arrow[r, "\text{induces}"] \arrow[d, "\partial"] & \text{Weinstein handlebody} \arrow[d] \\
\text{Dehn twist here} \arrow[r] & \text{Open book decomposition} \arrow[r] & \text{Contact surgery diagram}
\end{tikzcd}$$

Lefschetz fibrations:

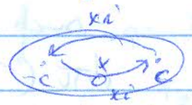


Critical points modeled on $(z_1, \dots, z_n) \mapsto z_1^2 + \dots + z_n^2$; crit pt @ 0.

Fibers over $c > 0$: $\begin{cases} \sum x_i^2 - \sum y_i^2 = c \\ \sum x_i y_i = 0 \end{cases} \rightsquigarrow T^*S^n$; 0-section is $\{y_1 = \dots = y_n = 0\}$

As we get closer to 0, the union of these spheres is a disc (thimble), and the sphere itself is a vanishing cycle.

Look at fiber over $c < 0$: same, exchange just x_i's and y_i's.
What if we turn around 0? We get monodromy in the fiber, which is a symplectomorphism, since we took a symplectic connection.

We started with $c > 0$; when we get to $c < 0$, we basically multiplied by i. Keep turning , we get -1, which is the antipodal map.

To see that we actually have a Dehn twist, we have to write it in coordinates.

Catherine Cannizzo - Lefschetz fibrations and open books.

Lefschetz fibrations:

A compact symplectic manifold (with corners) $(W, \omega = d\theta, \vartheta, I$ comp a-$\mathbb{C}$-s
A Lefschetz fibration $\pi : W^{2n+2} \to \mathbb{D}$ is a singular fibration, $d\pi \circ I = i \circ d\pi$, such that

① $\pi \pitchfork \partial\mathbb{D}$, $d\pi(T_x W) = T_{\pi(x)}\mathbb{D}$ for $\pi(x) \in \partial\mathbb{D}$, so no critical points on $\partial^v W = \pi^{-1}(\partial\mathbb{D})$.

② π is smooth on $\partial W \setminus \partial^v W = \partial^h W$, so no crit pts there

③ $\exists$ hol. charts around crit pts st π becomes $(z_1, \dots, z_{m+1}) \mapsto \sum z_i^2$.

To talk about parallel transport, we need a connexion

④ $T_x W = \underset{\ker d\pi_x}{\underset{\shortparallel}{T_x^v W}} \oplus \underset{(\ker d\pi_x)^{\perp\omega}}{\underset{\shortparallel}{T_x^h W}}$

⑤ $T_x^h W$ is tangent to horizontally boundary for $x \in \partial^h W$.

Rem: this implies that the fibers are symplectic.

This gives parallel transport maps away from critical values
$\beta : [0,1] \to \mathbb{D} \setminus \mathrm{critv}(\pi)$ induces $h_\beta : W_{\beta(0)} \to W_{\beta(1)}$

- Horizontal lift of $\beta'(t)$ to $p \in W_{\beta(0)}$ $\quad \beta(1) =: x_1$
- Integrating, get symplectomorphism $W_{\beta(0)} \to W_{\beta(1)}$.

Vanishing cycles: $0 \in \mathbb{D}$ regular value. Set $W_0 = \pi^{-1}(0)$.
A framing of Lagr sphere $L \subseteq W_0$ is a diffeo $\nu : S^n \to L$. Pick a path γ from 0 to some critical value. There exists a unique Lagrangian embedded ball which

- projects to γ
- $\partial\Delta_\gamma = L \subseteq W_0$

Result: Δ_γ is Lagrangian $\iff$ parallel transport maps $\Delta_\gamma \cap W_{\gamma(t)}$ to each other.

Framing: $B(T_{x_1}\Delta_\gamma) \longrightarrow \Delta_\gamma$: flow $\nabla(\gamma^{-1}\circ\pi|_{\Delta_\gamma})$, restrict to ∂.

A Lefschetz fibration determines the vanishing cycles + framings.

Definition: (abstract Lefschetz fibration) $W = (W_0, L_1, \dots, L_m)$ where W_0 is a Weinstein domain, and L_i are framed exact Lagrangian spheres.

How do we get a Lefschetz fibration $|W| \to \mathbb{D}$ from that?

- Equip $W_0 \times \mathbb{C}$ by the Liouville form on the fiber $\lambda_0 - J^* d(\frac{1}{2}|z|^2)$ and $\phi_0 + |z|^2$ a function.
- Lift L_j to Legendrian Λ_j of $(W_0 \times S^1, \lambda_0 + N d\theta)$ where each Λ_j lies over small interval of $\frac{2\pi j}{m}$ (N large so these intervals are disjoint)
- Embed $S^1 \hookrightarrow \mathbb{C}$ as circle of radius $\sqrt{N}$. Pulls back $-J^* d(\frac{1}{2}|z|^2)$ to $N d\theta$.
- $\Lambda_j \subseteq W_0 \times S^1$ on level set $\{|z| = \sqrt{N}\}$. Apply Liouville flow $\Lambda_j \times \mathbb{R}_{\geq 0} \to W_0 \times \mathbb{C}$.
- Flow of Λ_j will intersect $\{\phi_0 + |z|^2 = 0\}$ in Λ_j'.
- Attach Weinstein handles to Λ_j'.

Theorem (Giroux - Pardon): Let W be a Weinstein domain. Then $\exists$ an abstract Weinstein Lefschetz fibration $W' = (W_0, L_1, \dots, L_m)$ whose total space $|W'|$ is deformation equivalent to W.

Open book decompositions:

An abstract open book (AOB) is a Σ^{m-1} with boundary and a diffeo $\phi : \Sigma^{m-1} \to \Sigma^{m-1}$, which is the identity in a collar neighborhood of the boundary.

An open book decomposition (OBD) is an N^m with codimension 2 B^{2m-2} (binding) & a smooth fibration $p : N \setminus B \to S^1$ with $\partial p^{-1}(\theta) = B$. $p^{-1}(\theta)$ is a page

A Lefschetz fibration gives OBD/AOB.

AOB Take a fiber Σ over $pt \in \partial \mathbb{D}$.

Monodromy ϕ = product of Dehn twists about vanishing cycles

Attaching spheres in reverse order of product.

OBD M_ϕ = mapping tori of ϕ

$= \Sigma \times [0,1] / (x,0) \sim (\phi(x), 1)$

$= \partial^0 W = \pi^{-1}(\partial D)$

$\partial W = M_\phi \cup_{id_{\partial\Sigma \times S^1}} (\partial \Sigma \times \mathbb{D})$.

ex: $\overline{X}$ complex proj. variety, $\mathcal{L} \to X$ very ample line bundle. Take $s_0, s_\infty \in H^0(\mathcal{L})$ two sections, and look at their vanishing

Take $\mathbb{CP}^2$, $\mathcal{O}(2)$ $\quad s_0 = x_0(x_1 - x_2)$

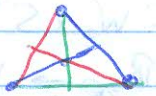
over $0, 1, \infty$ $\quad s_1 = x_1(x_2 - x_0)$

$s_0 + s_1 = x_2(x_1 - x_0)$

} vanishing gives 3 singular cones.

Remove base locus. Singular conics:

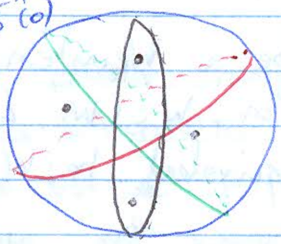
$\mathbb{CP}^2 \setminus B \to \mathbb{CP}^1$



To get Lefschetz fibration, remove vanishing of generic section

$\mathbb{CP}^2 \setminus \mathbb{CP}^1 \to \mathbb{C}$.

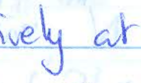
$T^* \overset{\shortparallel}{\mathbb{RP}^2} \quad \overset{\shortparallel}{s^{-1}(0)}$

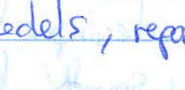


framed Lagr S^{n-1}

Proposition: Given $(W_0^{2n-2}, \{V_i\}_{i=1}^k)$ "abstract L.F.", $\exists$ $W^{2n} \xrightarrow{\pi} \mathbb{C}$ L.F. such that the vanishing cycles wrt a "distinguished basis of vanishing paths" are $\{V_i\}_{i=1}^k$, and then correspondingly a Weinstein manifold structure on def. equiv. to W. (*)

(*) $\leftrightarrow$





Proof: inductively at and paste local models, repaired so fibration genuinely trivial from, eg, $\sum |z_i|^2 \leq 1$.

Cédric De Groote - Legendrian contact homology I

Daniel Álvarez-Gavela - Legendrian contact homology II

1) Front and Lagrangien projection
2) Generating functions
3) Fronts of dim 1
4) Fronts of dim > 1
5) LCH in the front proj
6) An example

1)
$$\begin{tikzcd} & J^1(M,\mathbb{R}) = \overbrace{T^*M \times \mathbb{R}}^{dz - pdq} \arrow[dl] \arrow[dr] & \\ \underbrace{T^*M}_{dp\wedge dq} & & J^0(M,\mathbb{R}) = M \times \mathbb{R} \end{tikzcd}$$

Graphical submanifolds of T^*M = section $\alpha : M \to T^*M$ = 1-form

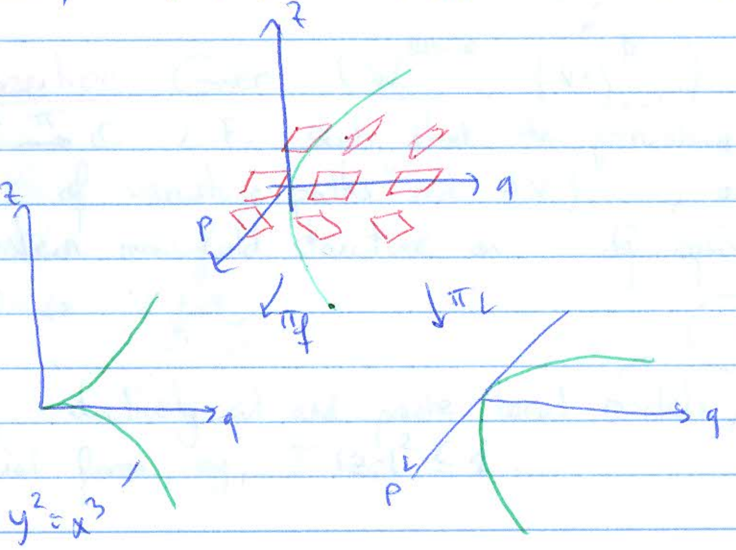
If is Lagr $\Leftrightarrow dp\wedge dq|_{\Gamma(\alpha)} = 0 \Leftrightarrow pdq$ is closed on $\Gamma(\alpha)$

$\Leftrightarrow \alpha^*(pdq)$ is closed $\Leftrightarrow \alpha$ is closed.

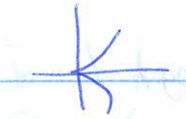
Definition: $\Gamma(\alpha)$ is exact if $\alpha = df$.

Recall we have Weinstein nbhd thm, so if 2 lagr are suff. close to each other, we can see one of them sitting in the cotangent bundle of the other one.

2) $M = \mathbb{R}$; $J^1(\mathbb{R};\mathbb{R}) = \mathbb{R}^3_{std} \rightsquigarrow \xi = \ker(dz - pdq) = \{\partial/\partial p, p\,\partial/\partial z + \partial/\partial q\}$



cusp $y^2 = x^3$



$F : M \times \mathbb{R}^N \to \mathbb{R}$

$\Sigma = \{ (q,x) \in M \times \mathbb{R}^N \mid F_x = 0 \}$

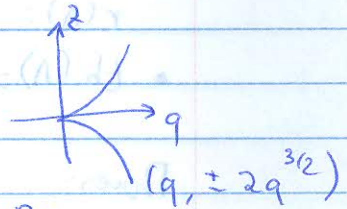
$\Lambda = \{ (F_q, F)(q,x) \in \underbrace{T^* M \times \mathbb{R}}_{J^1(M;\mathbb{R})} , (q,x) \in \Sigma \}$

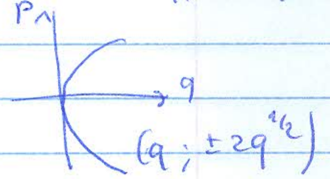
$\to F(q,x) = x^3 - 3qx$

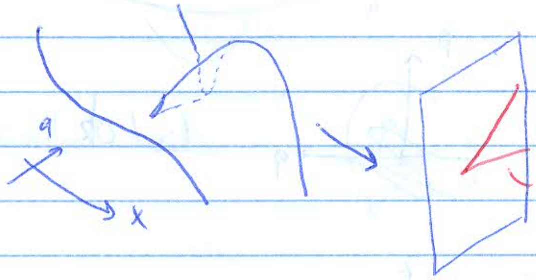
$F_x = 3x^2 - 3q \longrightarrow \Sigma = \{ q = x^2 \}$

$F_q = -3x = \pm 3 q^{1/2}$

$F = \pm 2 q^{3/2}$

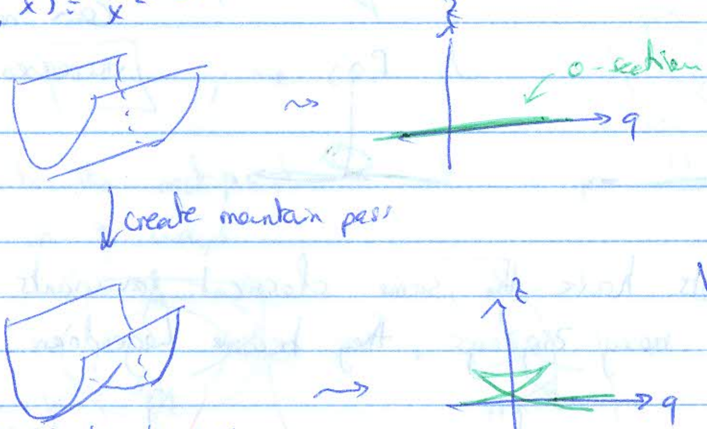




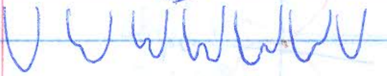


Families of Legendrian are also important.

$F(q,x) = x^2$

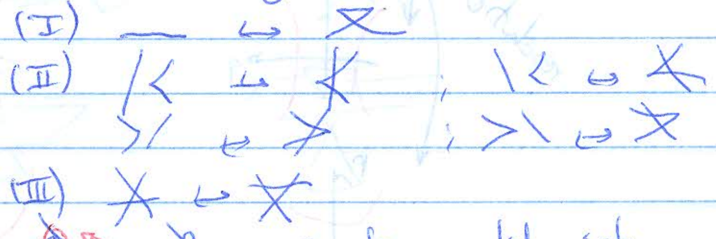


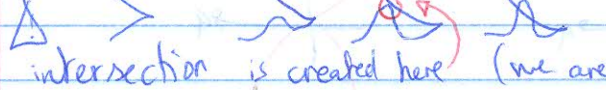
We see the 1st Reidemeister move appear



3) $S^1 \overset{\Lambda}{\hookrightarrow} J^1(\mathbb{R}, \mathbb{R})$ Legendrian knot.

Reidemeister moves:



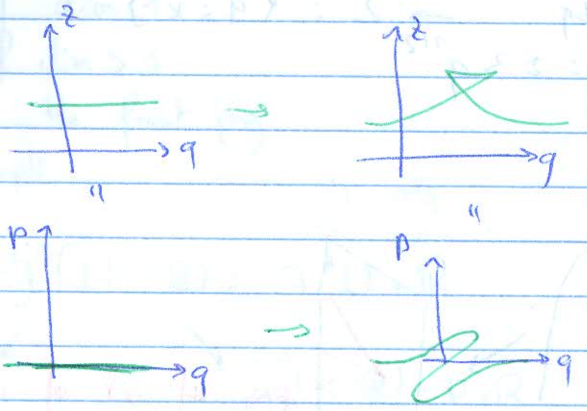


is not a valid isotopy, since a self intersection is created here (we are in front projections)

Classical invariants (of Leg. knots up to Leg. isotopy)

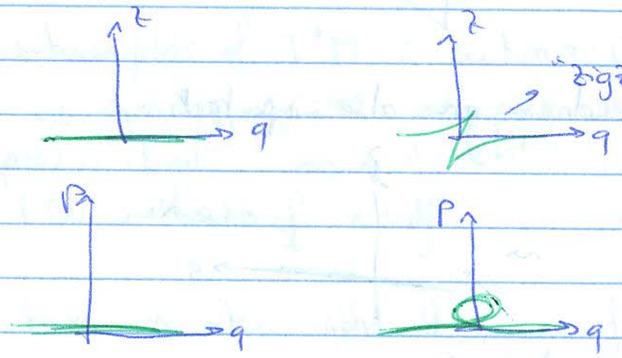
- smooth isotopy type
- $r(\Lambda)$, $m(\Lambda) = 2r(\Lambda)$
  $r(\Lambda) = \deg(\sigma(\pi_L(\Lambda)) : S^1 \to S^1)$
- $tb(\Lambda) = \mathrm{link}(\Lambda, \tilde{\Lambda})$; $\tilde{\Lambda} = \Lambda$ pushed off in $\partial/\partial z$ direction

Moves:



ok

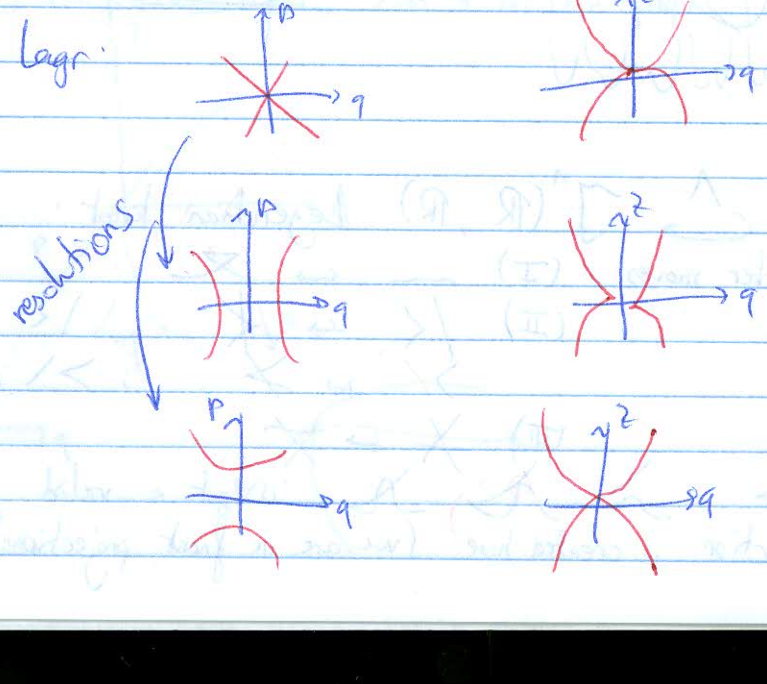
or



can not happen, as it increases the rotation number.

Rem: if 2 knots have the same classical invariants, then after adding sufficiently many zigzags, they become Legendrian isotopic.

Resolution of Lagr:



4) In higher dimensions

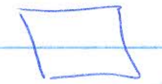
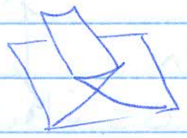
Observation: any front trivially multiplied by $\mathbb{R}^N$ gives a front in higher dimension. We do not have to multiply by a straight line; we can do it with eg a circle.

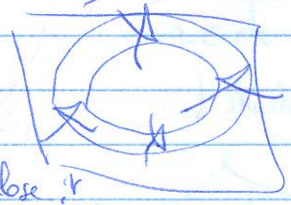
ex.  $= > \times \mathbb{R}$

 $= > \times S^1$

Swallow tail: 

So "multiplying" also works in families.

From , we can do , but this is not relative to the rest: it is not id on ∂ of the square. We can make it in 3 ways:

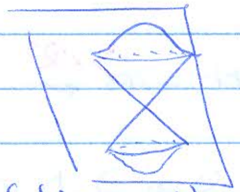


(close it in circle)

or

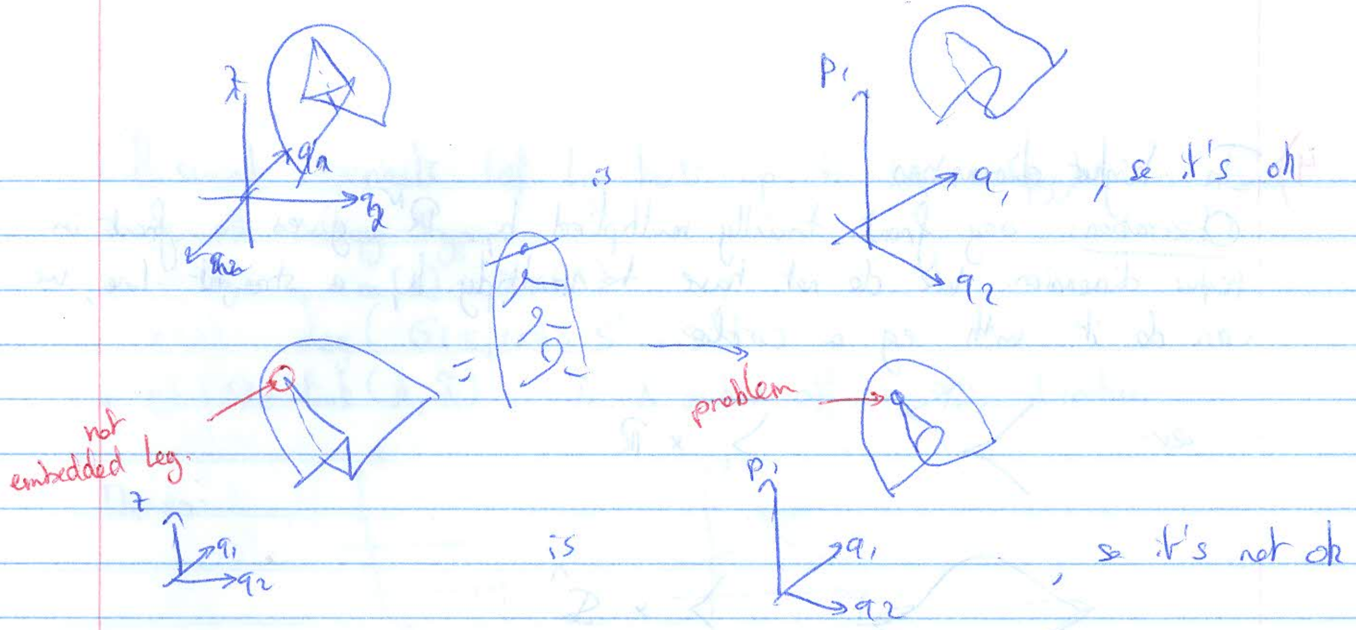
(make it die)

or

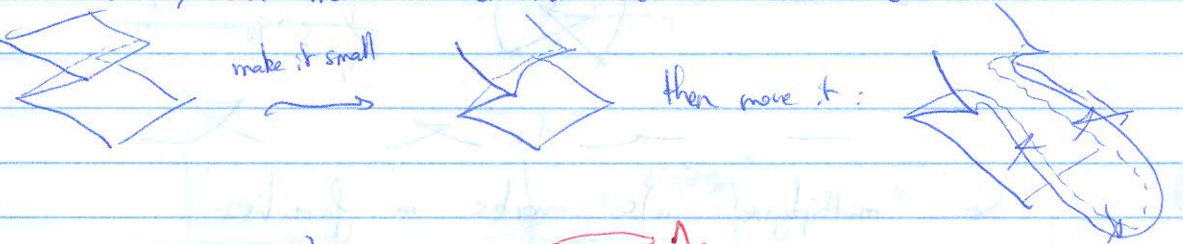


(times S^1)

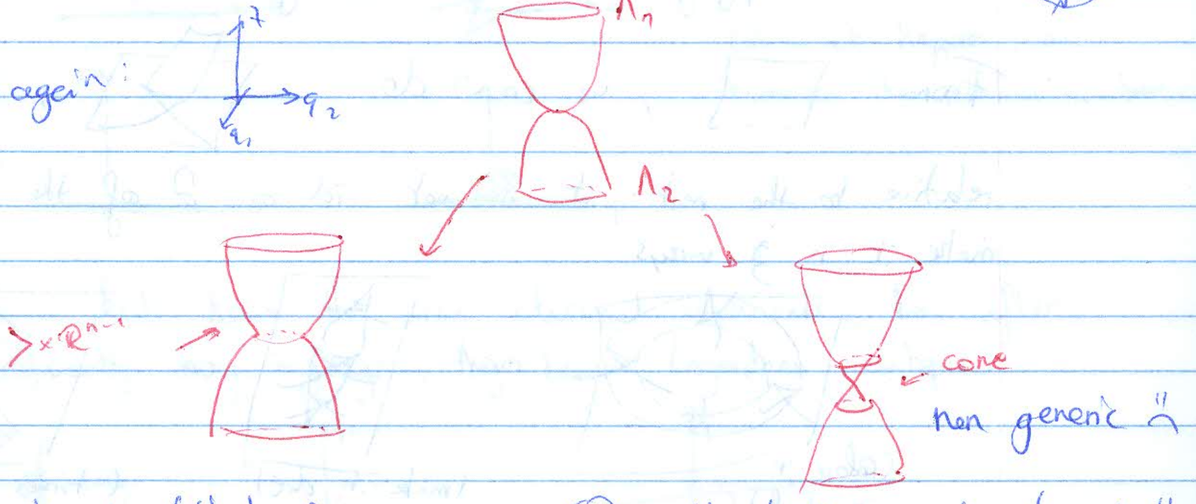
Possible moves: 



In higher dimension, 1 zigzag is enough to get flexibility: we can make it small, and move it around in the extra dimension.

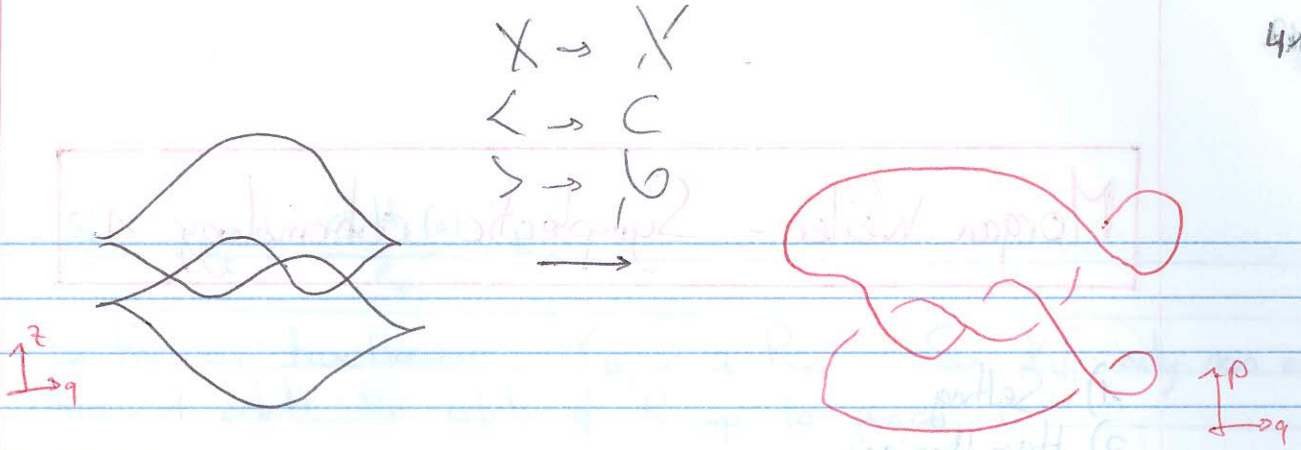


Surgery again:



We can resolve it (think of cone as ○ with light emanating towards the inside, seen in space time; change circle ○ into ellipse ⬭).

5)



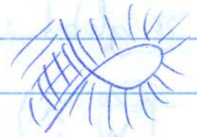
So A_{LCH} = free assoc DGA /$\mathbb{Z}_2$ generated by X and >.

So we see  or  for the + crossing; NEVER 

There is a list of things we can get, and of things we can't.

Prohibited: , 

Not prohibited but don't contribute to ∂:

   (⊂  is lagr)

Proposition:  ⊆ ∧ ⇒ $LCH_* = 0$.


a∈A

∂a = 1+.. cf -action formula

?

↓ shrink it, we can assume a has action as small as we want ⤳ ∂a = 1+0


a∈A

⇒ 1 ∈ im ∂

⇒ $LCH_* = 0$.

In higher dimensions, we can make  thin; same idea.

Morgan Weiler - Symplectic cohomology 1:

1) Setting
2) Hamiltonians
3) $\mathcal{A}$, generators, Λ_ω, 1:1
4) Moduli spaces, ∂
5) Maximum principle + continuation maps

1) Setting:

$\mathcal{L}_X \omega = \omega$

(W^{2n}, ω) compact symplectic, of contact type: $\exists$ Liouville v.f. X defined near ∂W, $X \pitchfork \partial W$, outwards pointing

Claim: $\lambda = i_X \omega$ has $d\lambda = \omega$

$\alpha := \lambda|_{\partial W}$ is contact, determines $(M = \partial W, \xi)$

(W, ω) is a Liouville domain if X is global ($\Leftrightarrow \lambda$ is global)

SH is an invariant of the completion $(\hat{W}, \hat{\omega}) = (W \cup M \times \mathbb{R}_{\geq 0}, \omega \cup d(e^a \alpha))$.
To know that this is symplectic, check that there is a neighbourhood of ∂W which is symplectomorphic to $(M \times (-\delta, 0], d(e^a \alpha))$

map: $G: (m, a) \longmapsto \phi_X^a(m)$

Get $G^* \lambda_{(m,a)} = e^a \alpha$, & $G^* \omega_{(m,a)} = d(e^a \alpha)$

2) Hamiltonians:

$H: S^1 \times \hat{W} \to \mathbb{R}$ st
- time 1 orbits of X_{H_t} are non-degenerate

(• things about C^2 small, to control Morse homology in W)

- $H(\theta, m, a) = \tau e^a + c$ when $a \geq T$ in symplectization part, for $\tau \notin \mathrm{Spec}(M, \alpha) = \{$periods of Reeb orbits of $\alpha\}$

↳ this set is discrete, so τ exists

Why? What is X_{H_t} in cylindrical end?

$\hat{\omega}_{(m,a)}(\partial_a H_t(a) R_m, \cdot)$ (R_m = Reeb vf of α at m)

$$= e^a da \wedge \alpha_m (\partial_a H_t(a) R_m, -) + e^a \underbrace{d\alpha_m(\partial_a H_t(a) R_m, -)}_{=0 \text{ because Reeb}}$$

$= \partial_a H_t(a) e^a da$

$= e^a dH_t$

So, $X_{H_t}^{(m,a)} = \dfrac{\partial_a H_t(a)}{e_a} \cdot R_m$.

↳ For our hamiltonians, $X_{H_t} = \tau R_m$. So, X_{H_t} only sees as time-1 orbits the orbits of R up to period τ.

3) A, generators, Λ_m, 1.1.

Fix a representative h of each free loop space class of W.
Considers pairs $(\gamma, [\sigma])$ where σ is a map $\Sigma \to W$ realizing that γ and h are homologous.
Or, for simplicity, assume W is exact. Or fix a repr of the 0 class of W; consider pairs $(\gamma, [\sigma])$ where σ is a map $\mathbb{D} \to W$ filling γ, so $\sigma|_{\partial\mathbb{D}} = \gamma$.

Almost-$\mathbb{C}$-str:
- J compatible with $\hat{\omega}$
- In the end $\Gamma \times [T, \infty)$,
  1) J invariant under $\mathbb{R}$ translation
  2) $J\partial_a = R$
  3) $J\xi = \xi$; J comp with $d\alpha$.

We want to consider the gradient flow of
$$A_{H_t}(\gamma) = -\int_{\mathbb{D}} \sigma^*\omega \textcircled{$\mp$} \int_0^1 H_t(\gamma(t))\,dt$$
↳ this sign differs in different places/books

$$\to d_{(\gamma,[\sigma])} A_{H_t}^{(\cdot)} = \int_0^1 \hat{\omega}(\dot{\gamma}(t) + X_{H_t}, \cdot)\,dt$$

The J gives metric on $\mathcal{L}\hat{W}$: $\int_0^1 \hat{\omega}(\cdot, J\cdot)\,dt$.

$$\to \nabla_{(\gamma,[\sigma])} A_{H_t} = -J(\gamma(t))(\dot{\gamma}(t) + X_{H_t}).$$

What exactly is the domain of the action functional?
$A_{H_t} : \{(\gamma, [\sigma]) \mid \gamma \in \mathcal{L}W, \text{ nullhomotopic}, [\sigma] \in H_2(W) \text{ has } \partial\sigma = \gamma\} / \sim$,
where $[\sigma] \sim [\sigma']$ if $\int_{\mathbb{D}} \sigma^*\omega = \int_{\mathbb{D}} \sigma'^*\omega$.

u is a "gradient flow" of ∇A_{H_t} if $\partial_s u + J(u)(\partial_t u + X_{H_t}) = 0$

Critical points of A_{H_t} $\leftrightarrow$ zeroes of ∇A_{H_t} $\leftrightarrow$ $\dot{\gamma}(t) = -X_{H_t}$.

Grading? Choose a symplectic trivialization of $\sigma^* T\hat{W}$; we can compute $\mu_{CZ}(\gamma)$ wrt this trivialization. Set $|\gamma| = n - \mu_{CZ}(\gamma)$.

$\rightsquigarrow$ $SC^*(H_t)$ is generated (over $\mathbb{Z}$?) by 1-periodic orbits of $-X_{H_t}$.

4) Moduli spaces:

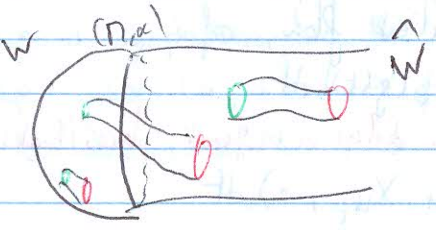
$\mathcal{M}^A(\gamma_-, \gamma_+, H_t, J)$ = solutions $u: \mathbb{R}\times S^1 \to \hat{W}$ to the equation $\partial_s u + J(u)(\partial_t u - X_{H_t}) = 0$, with limit at $+\infty$ to γ_+, $-\infty$ to γ_-

and we want $[\sigma_{\gamma_-}] + [u] = [\sigma_{\gamma_+}] + A$,
all of this modded out by $\mathbb{R}$.

$$\rightsquigarrow \partial\gamma_+ = \sum_{\gamma_-, A} \sum_{u \in \mathcal{M}^A(\ldots)} \varepsilon(u)\gamma_-$$

$\dim \mathcal{M}^A(\gamma_-, \gamma_+, H_t, J) = 0$

⚠ according to the signs we are using, we go from $+\infty$ to $-\infty$, or from $-\infty$ to $+\infty$.



What do we need for compactness?
1) Energy bound: $E(u) = A_{H_t}(\gamma_-) - A_{H_t}(\gamma_+) > 0$
2) Need to know that the u's can't escape to ∞.
Solution: find a way to argue that $u \in \mathcal{M}(\ldots)$ is contained in a compact region in $\hat{W}$, depending only on γ_- and γ_+.

$SH^*(W,\omega) \rightsquigarrow \varinjlim SH^*(H_t)$

5) C^0 Maximum principle: if u is a solution to $\partial_s u + J(u)(\partial_t u + X_{H_t}) = 0$ and u goes into symplectization, consider $u = (v$ on M, f on $\mathbb{R})$.
Goal: subharmonicity $\Delta f \geq 0$, so max of f is attained on $\partial(\mathbb{R}\times S^1)$.
$\hookrightarrow$ get it from splitting C-R equations in 3 directions.

Say I have 1-param family of Hamiltonians $H_{t,s}$, do computations to show $\Delta f \geq 0$; get $\Delta f + \partial_s \tau_s \geq 0$ $\rightsquigarrow$ need $\partial_s \tau_s \leq 0$ for moduli $\mathcal{M}^A(\gamma_-, \gamma_+, H_{t,s}, J)$ to be compact
(to def. continuation maps $SC^*(H_{t,1}) \to SC^*(H_{t,0})$), need eventual slope to be decreasing, so only get cont.-map when $\tau_0 \geq \tau$

τ_s = eventual slope of $H_{t,s}$
⊕ counting $\gamma_\pm$ γ_0 $H_{t,p}$ $H_{t,1}$

Umut Varolgunes - A_n diagrams

Recall: an abstract Weinstein Lefschetz fibration (AWLF) is the data of

- a Weinstein domain M
- a linearly ordered collection of exact framed lagr spheres inside M $L_1 \dots L_k$

This gives $E \to \mathbb{D}$ a lefschetz fibration, with E Weinstein domain

vanishing cycle is L_i

E here is well defined only up to deformation equiv. of Weinstein domains, like everything else in this talk.

We are interested in special kinds of AWLF's:

- $M = A_n$-Milnor fibre, ie $X_n^d = \{p(x) = z_1^2 + \dots + z_d^2\} \subseteq \mathbb{C}^{d+1}$, where $p(x)$ is a polynomial with simple roots, of degree $n+1$.

X_n^d

$\big\downarrow$ x coordinate Lefschetz fibration

Note that $X_1^d \simeq T^*S^d$.

critical values at the roots of p. →

← regular fiber $= \{z_1^2 + \dots + z_d^2 = s\}$ $= T^*S^{d-1}$

Let's talk about Lagrangians in there.

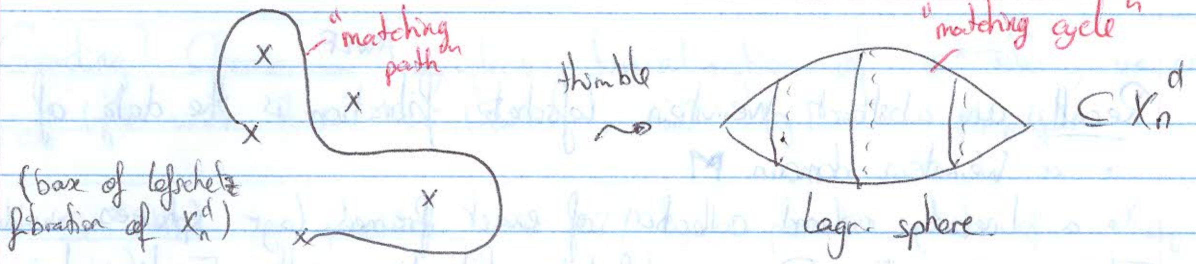
* Inside each regular fiber, there exists a canonical framed exact Lagrangian sphere.

- If s is real: real locus $\subseteq \{z_1^2 + \dots = s\}$, ie 0-section of T^*S^{d-1}
- For general s: choose $\sqrt{s}$, take the image of

$$\{z_1^2 + \dots + z_d^2 = 1\} \xrightarrow[\cdot\sqrt{s}]{\cong} \{z_1^2 + \dots + z_d^2 = s\}$$

These Lagrangian spheres are the vanishing cycles associated to all critical values; in particular, they are parallel transport of each other, using any path.

Lagrangians in X_n^d are required to be matching cycles:



Definition: an A_n-diagram is the following data:

- $n+1$ distinct points in the plane (these crosses)
- an ordered collection of matching paths
- the dimension d.

The set of A_n-diagrams has a natural topology induced from $\mathbb{C}$, and a continuous path in that topological space is called an isotopy of A_n-diagrams

$$\{A_n\text{-diagrams}\} \xrightarrow{\text{"}\{AWLFS\}\text{"}} \{\text{Weinstein manifolds}\}:$$

- $n+1$ points in the plane is to be considered as one of the standard Lefschetz fibrations of X_n^d (choose $p(x)$ that has these roots)
- matching paths $\rightsquigarrow$ matching cycles $\subseteq X_n^d$.

Now, ~~let~~ it descends to $W: \{A_n\text{-diagrams}\}/\text{isotopy} \longrightarrow \{\text{Weinstein manifolds}\}/\text{def. equivalence}$

Properties of W:

1) It is not injective. Recall that $\{AWLFS\}/_{\text{isotopy}} \to \{\text{Weinstein}\}/_{\text{def equiv}}$ is also not injective?:
   (1) Stabilization (increases # of Lagrangians)
   (2) Hurwitz moves
   (3) Cyclic shifts in the linear order

Which of these can we make in the A_n-diagrams? (2) and (3) can always be done, and certain kinds of (1) too.

2) It is not surjective: there are obvious topological restrictions for $\dim \geq 4$: the H_1 of what we construct is going to be 0. exercise in dim 4

More on W tomorrow by Roger. Today we will try to go in the other direction, ie find some Weinstein manifolds which are in the image of W, and explicitely give A_n-diagrams associated to them.

Strategy (obvious): "it is expected that all exact Lagrangian spheres in X_n^d are isotopic to matching cycles." (heuristic)
Hence, we take our manifold X.

- try to find a Lefschetz fibration with X_n^d as its regular fiber.
- find matching cycles which are isotopic to the vanishing cycles at a fixed regular fiber.

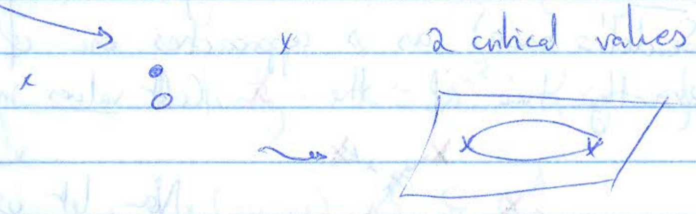
Example: $\mathbb{C}^{d+1}$?

- Lefschetz fibration: $$\begin{tikzcd} \mathbb{C}^{d+1} \arrow[d, "z_1^2+\dots+z_{d+1}^2"] \\ \mathbb{C} \end{tikzcd}$$
- 0 is a critical value, fiber above 1 is T^*S^d. We want to find the ~~matching~~ vanishing cycle from 0 to 1. It has to be the real part of T^*S^d, ie the 0-section.

For the matching cycle, project $$\begin{tikzcd} \{z_1^2+\dots+z_{d+1}^2=1\} \arrow[d, "z_1\text{- coordinate}"] \\ \mathbb{C} \end{tikzcd}$$

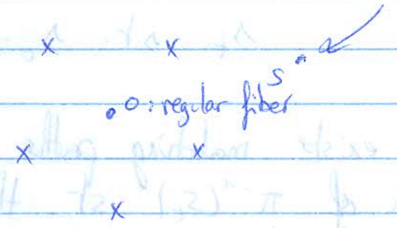
Example: T^*S^{d+1}? $= \{z_1^2+\dots+z_{d+2}^2=1\}$
$$\begin{tikzcd} {} \arrow[d, "z_1"] \\ {} \end{tikzcd}$$

$X_1^d \cong T^*S^d$



Harder example: $X = \{x^2y^3-1=z^2\} \subset \mathbb{C}^3$.
$$\begin{tikzcd} {} \arrow[d, "2x+3y"] \\ \mathbb{C} \end{tikzcd}$$

means $\begin{cases} 2x+3y=s \\ x^2y^3-1=z^2 \end{cases}$
$\wr$
5^{th} deg pol in x
$\wr$
X_4^1

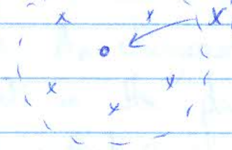


We want to find the vanishing cycles corresp to the crit pts above 0, ie in the regular fiber, and see them as matching cycles in the Milnor fiber.

Idea: use a Lefschetz bifibration,

$$\begin{tikzcd} X \arrow[d, "{(2x+3y, x)}"] \arrow[dd, bend right, "2x+3y"'] \\ \mathbb{C}^2 \arrow[d, "pr_1"] \\ \mathbb{C} \end{tikzcd}$$

$\mathbb{C} \supseteq$



Let s be a regular value of the 1st Lefschetz fibration ($X \to \mathbb{C} : 2x+3y$).

For each value of s, which is not a critical value of π, there exists a secondary Lefschetz fibration

$$\begin{tikzcd} \pi^{-1}(S) \arrow[d, "\text{project to x coord}"] \\ \{s\} \times \mathbb{C} \end{tikzcd}$$

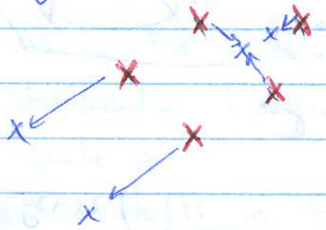
from the map $\begin{tikzcd} X \arrow[d] \\ \mathbb{C}^2 \end{tikzcd}$.

Let us use $s=0$ as the base point in the base of the secondary Lefschetz fibration of $s=0$; we see

As we move s, the critical values of the secondary Lefschetz fibration move in the plane (it's hard/impossible to solve for critical values parametrically for s~~t~~; imagine doing very precise approximations, using computers).

see these things with a computer

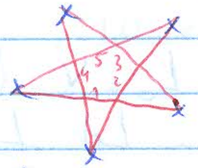
Fact: since we have our fibrations generically (whatever that means – see Paul Seidel's book) as s approaches one of the critical values of π, exactly two of the x (crit values in the fiber) will come together:



Now, let us take a matching cycle in $\pi^{-1}(s)$ wrt the secondary Lefschetz fibration, and see what happens to it when we parallel transport it to another regular fiber $\pi^{-1}(s')$ along some path s_t s.t. $s_0 = s$, $s_1 = s'$.

Fact: if there exists matching paths l_t inside the base of the secondary Lefschetz fibration of $\pi^{-1}(S_t)$ st the A_n-diagrams
(crit values of 2ndary LF of $\pi^{-1}(S_t)$, l_t)
form an isotopy of A_n-diagrams,
Then the matching cycle of l_1 is isotopic to the parallel transport of the matching cycle of l_0

Now, suppose that s becomes close to one of the critical points: 2 red points collide, and their matching cycle (just before they collide) has to be the vanishing cycle of s before s hits the critical value.

In the end we get the following: 

, by transporting back to the origin.

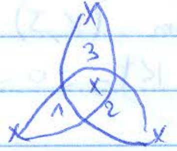
Rem: 1) Is this a proof? No: the computer can miss small steps. We can turn it into a proof by estimating errors via bounds on derivatives, ...

2) This method works for any $p(x,y)$ in place of x^2y^3-1 (as long as the solution set is smooth).

3) Dimension: adding more z variables, ie $\{p(x,y)=z_1^2+\dots+z_d^2\}$, we get exactly the same diagram (just with different specified dimension). It just changes T^*S^1 to T^*S^d.

4) We actually get a cyclic order because we chose the origin as the basepoint, instead of a linear order.

One more example: $p(x,y)=x(xy^2-1) \rightsquigarrow X=\{x(xy^2-1)=z^2\}$

$\rightsquigarrow$ get 

Actually: $X \simeq \mathbb{C}^2 \setminus \{xy=1\}$.

$\{x(xy-1)=1\} \subseteq \mathbb{C}^3$.

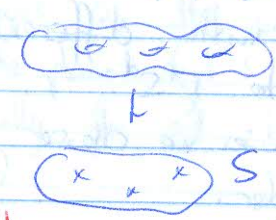
Netanel Blaier – Lagrangian Floer homology and the Dehn twist long exact sequence

Let $L_0 = V$ be an exact ~~be an exact~~, framed, Lagrangian sphere. Let L_1, Q be another exact Lagrangian. Then,

$$\begin{tikzcd}
HF(V, L_1) \otimes HF(Q, V) \arrow[rr] & & HF(Q, L_1) \arrow[dl] \\
& HF(Q, \tau_V L_1) \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

The Floer TQFT (1):

Let $\pi : (E, \omega_E, J_E) \to (S, \omega_S, J_S)$ be a LF:



$\partial^v E = \pi^{-1}(\partial S)$

$\partial^h E = \partial E \setminus \partial^v E$

Definition: a (moving) Lagrangian boundary condition is a family $F_z \subseteq E_z$ where $z \in \partial S$.

Definition: a relative perturbation datum (K, J) is (fix F)

- $K \in \Omega^1(E)$, $K|_{TE^v} \equiv 0$, $K|_F = 0$ in $\Omega^1(F)$.
- $K \equiv 0$ near $\partial^h E \cup \mathrm{crit}(\pi)$
- J ω_E-comp almost-$\mathbb{C}$-str such that π is J-holo and $J = J_E$ near $\partial^h E \cup \mathrm{crit}(\pi)$.

Definition: an inhomogeneous section with boundary in F is: $s : S \to E$ s.h

(1) $\pi(u(z)) = z$

(2) Elliptic PDE: $(Du - Y(u))^{0,1} = 0$

where $Y \in C^\infty(E, \mathrm{Hom}(\pi^* TS, TE^v))$ does the following things: given any $\xi \in (TS)_{\pi(x)}$, we set $Y(\xi)$ = Hamiltonian v.f. for E_z, for the Hamiltonian $K(x)$, which does not depend on the choice of lift of x

If $\dim M = 0$, then $\Phi_{E/S} = \# M_{E/S} \in \mathbb{Z}_2$

($k(x)$ some cutoff fct)

Example (local model) $E = \{ x \in \mathbb{C}^{n+1} \mid \begin{smallmatrix} Q(x) \leq r \\ (k(x) \leq s) \end{smallmatrix} \}$ ↳ not important

$\downarrow (x_1, \dots x_n) \mapsto \sum x_i^2$

$S = \mathbb{D}^2$

- The Lagr ∂ condition is $F_z = \sqrt{z}\, S^n$, $|z| = r$
- Rel perturbation data : $(k, J) = (0, J_E)$
- Pseudo-hol sections :

  ② ③ $\Rightarrow u_a : S \to E$, $u_a = \frac{1}{\sqrt{r}} a z + \sqrt{r}\, \bar{a}$, $a \in \mathbb{C}^{n+1}$

  ① $\Rightarrow a_1^2 + \dots + a_{n+1}^2 = 0$, $\|a\|^2 = 1/2$
- $\dim \mathcal{M}_{E/S} = \cancel{2}\ 2n-1 \Rightarrow \Phi = 0$.

The Floer TQFT (2):

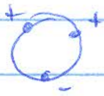
incoming (output) outgoing (input) ⟵ BAD IDEA

We study base $\hat{S}$; $\Sigma = \Sigma^- \cup \Sigma^+$

around them, fix hol. coordinates : strip-like ends.

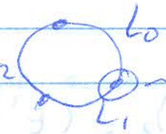
ex: 

= strip 



= 

for ξ, we know what Lagr are around it.

Now, think about Lagrangian labels : 

$\rightsquigarrow (L_{\xi,0}, L_{\xi,1})$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{L_0} \quad \overset{\shortparallel}{L_1}$

Seidel p239

For every L_0, L_1, we choose Floer data $(H, 0)$

~~Pf~~ with strip-like ends :

$$\begin{tikzcd} E \arrow[d] & Z^{\pm} \times M \arrow[l] \arrow[d] \\ S \supseteq u_\xi & \cancel{\quad} \ \mathbb{R}^{\pm} \times (\cdot, 1) =: Z^{\pm} \arrow[l, "\sim"', "\varepsilon_\xi"] \end{tikzcd}$$

(trivializing bundle over strip-like ends)

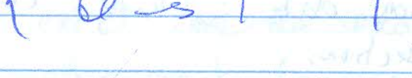
There is a corresponding notion of Lagr boundary condition :
they become constant

$\forall \xi \in \Sigma$, $\exists (L_{\xi,0}, L_{\xi,1})$ st

$F_\xi(s,k) = L_{\xi,k}$

Let L_0, L_1, $\varphi(L_0, L_1) = \left\{ \begin{array}{l} y : [0,1] \to M \\ y(0) \in L_0 \\ y(1) \in L_1 \\ \frac{dy}{dt} = \underset{\uparrow \text{ han vf of } \Omega}{X(t, y(t))} \end{array} \right\}$

Definition: $CF(L_0, L_1) = \mathbb{Z}_2$-vs freely generated by $\varphi(L_0, L_1)$.

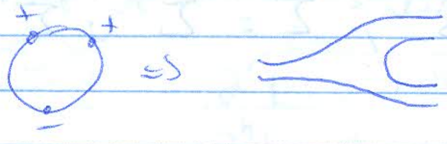
$\leadsto \mathcal{M}_{E/S}(\{y_\xi\}) = \{$  $\}$

Let $\Phi_{E/S} : \bigotimes_{\xi^+ \in \Sigma^+} CF(L_{\xi,0}, L_{\xi,1}) \to \bigotimes_{\xi \in \Sigma^-} CF(L_{\xi^-,0}, L_{\xi,1})$

mapping $\bigotimes_{\xi^+} y_{\xi^+} \mapsto \sum \# \mathcal{M}^0_{E/S}(\{y_{\xi^-}, y_{\xi^+}\})$

In particular,

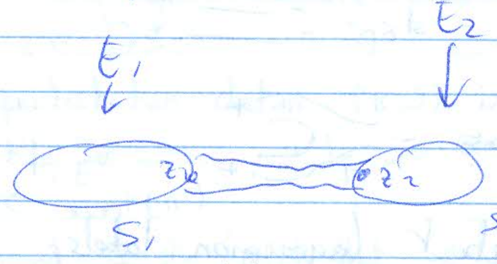
$= \mu^1$



$= \mu^2$

$\Rightarrow$ The cohomology of this complex wrt μ^1 $H(CF(L_0, L_1)) = HF(L_0, L_1)$, and μ^2 descends to a product on $HF(L_0, L_1)$.

Gluing theorem:



st $(E^1)_{z_1} = (E^2)_{z_2} \cong M$ $(F^1)_{z_1} = (F_2)_{z_2} \cong L$ $\leadsto$ get $\begin{array}{c} E \\ \downarrow \\ S \end{array}$

In particular,

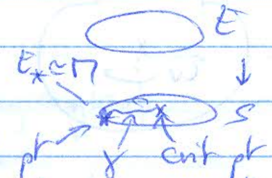
Theorem [Seidel]: $\mathcal{M}_{E/S}(\{y_{\xi^+}, y_{\xi^-}\})^0 = \coprod_{p+q=n} (\mathcal{M}_{E_1/S_1})^p \times_{L} (\mathcal{M}_{E_2/S_2})^q$ $\uparrow$ fiber product

Lemma: $\Phi_{E/S} = 0$, $f_2 = V$

Corollary: if $\Phi_{E_2/S_2, z_2} = 0 \in \overset{\overset{HF(L,L)}{\shortparallel}}{H^*(L)}$, then $\Phi_{E/S} = 0$.

The exact triangle (1)

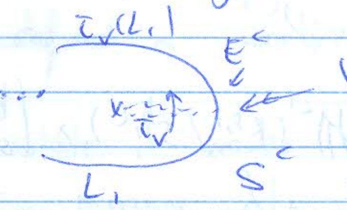
Let $L_0 = V$ exact, framed Lagrangian. We fix Lagr L_1 and $L_2 = \tau_V L_1$ in M.

Lemma: $\exists$ LF $E \xrightarrow{\pi} S$ st  st $\overset{\text{vanishing cycle of } \gamma}{v_\gamma} = V$

(one marked pt, one crit pt).

Step 1: define a Floer cocycle $c \in CF(L_1, L_2)$; $\mu^1(c) = 0$.

Remove $*$:



We cut it there for mental convenience: it's as if we glued the 2 parts with a Dehn twist.

Take $c = \Phi_{E^c/S^c}$

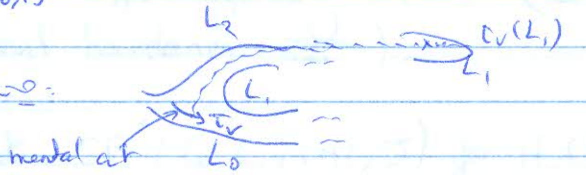
$\mu^1 \circ c = 0$:  $= \partial\left(\begin{smallmatrix}\text{1-dim mod} \\ \text{space with boundary}\end{smallmatrix}\right)$ = even number of pts $= 0$ in $\mathbb{Z}_2$-alg.

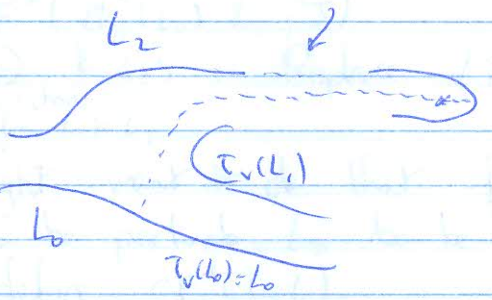
Step 2: $K : CF(L_0, L_1) \to CF(L_0, L_2)$

$$\mu^1(k(\cdot)) + k(\mu^1(\cdot)) = 0$$

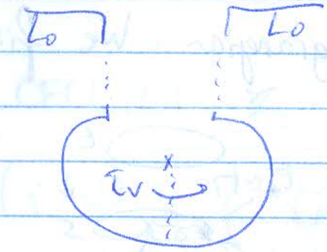
$$\mu^2(c, \cdot) = 0$$

The homotopy comes from counting sections in a 1-param family of LF's $\begin{matrix} E^{k,t} \\ \downarrow \\ S^{k,t} \end{matrix}$ $t \in [0,1)$

$r \to \infty$:  $\mu^2(c, \cdot)$



<u>r=1:</u> $\underline{L_2 \simeq \tau_V(L_1)}$



Let k count 1-sections in the family $E^{k,t} \to S^{k,t}$.
Then $\mathcal{M}^0(E^{k,t}/S^{k,t})$ is a 1-manifold $\Rightarrow \# \partial \mathcal{M}^0 = 0$.

- $\mu^0(E^{k,0}/S^{k,0}) = \mu^2(c, \cdot)$
- $\mu^0(E^{k,1}/S^{k,1}) = 0$
- broken things $\coprod_{r \in (0,1)} \mathcal{M}^{-1}(E^{k,t}/S^{k,t}) \times \left(\begin{smallmatrix}\text{broken}\\\text{strips}\end{smallmatrix}\right) \Rightarrow \mu^1(k(\cdot)) + k(\mu^1(\cdot))$. $\square$

<u>Step 3</u>: let Q be an axiliary Lagrangian
We want $Q: CF(Q, L_1) \to CF(Q, L_2)$
$e: CF(Q, L$ $(\cdot \cdot)$

Punchline: we show that the twist functor <u>is</u> the Dehn twist.

Sheel Ganatra's interlude: Lagrangian Floer homology & the Fukaya category (impressionistically):

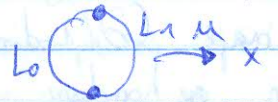
Let $L_0, L_1 \subseteq (X^{2n}, \omega)$ Lagrangians. $\rightsquigarrow HF^*(L_0, L_1)$, categorifying intersection number.

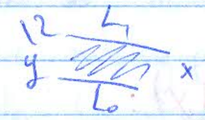
Properties:
- $\chi(HF^*(L_0,L_1)) = L_0 \cdot L_1$ (intersection number)
- If $L_0 \cap L_1 = \phi$, then $HF^*(L_0, L_1)$
- Hamiltonian isotopy invariant: $L_0 \underset{Ham}{\sim} L_0' \Rightarrow HF^*(L_0,L_1) \cong HF^*(L_0', L_1$

Not always well defined !! analysis / obstructions.

Simplest case where it works: $(X^{2n}, \omega = d\lambda)$ Liouville (e.g. Weinstein), $L_i \subseteq X^{2n}$ exact, meaning $\lambda|_{L_i} = df_{L_i}$, $f_{L_i} : L_i \to \mathbb{R}$.

Definition: Assume $L_0 \pitchfork L_1$. $\underset{\circlearrowleft d = \mu^1}{CF^*(L_0, L_1 \overset{;J}{\vee})} := \bigoplus_{x \in L_0 \pitchfork L_1} \Bbbk\langle x \rangle$.

It is d that depends on J. And $dx = \sum n_{xy} \cdot y$, where n_{xy} is the number of maps  , with $\overline{\partial}_J u = 0$, mod. reparam.



Rem: in good case (like this case), d is well-defined (for generic J), and $d^2 = 0$ (usual Morse-type argument: $\partial\{\overline{\underline{I}}\} = \{\overline{\underline{I}}\} \times \{\overline{\underline{I}}\}$. (obstructed case: additional boundaries ).

Variants: we can look at $CF^*(L_0, L_1; (H), J)$ for H (time-dep) Hamiltonia
(1) $CF^*(\phi^1_H(L_0), L_1; J)$
or (2) $\bigoplus_{\substack{x \text{ time 1 chords} \\ \text{of } X_H \text{ from } L_0 \text{ to } L_1}} \Bbbk\langle x \rangle$. Replace $\{\overline{\partial}_J u = 0\}$ by $\{\overline{\partial}_J^{X_H} u = 0\}$.

We might want to do that if L_0 is not transversal to L_1; then it allows to perturb L_0 to be transverse to L_1.
ex: define $CF^*(L, L)$.

The result is independant of H, J generic, & satisfies properties.

Rem: to get compactness of moduli spaces in X (non-compact Liouville manifold), prescribe a form of J near ∞, so that we get a maximum principle for the u's (cf Morgan's talk).

More structure: "multiplication maps"

$CF(L) := CF(L,L)$

$\mu^2 : CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \longrightarrow CF^*(L_0, L_2)$.

Special case: $L_0 = L_1 = L_2 =: L$: $\mu^2 : CF^\circ(L)^{\otimes 2} \to CF^\circ(L)$.

On $HF^*(L) \simeq H^*(L)$ (PSS), this recovers the cup product.

$x \in L_1 \pitchfork L_2$
$y \in L_0 \pitchfork L_1$

$\mu^2(x,y) = \sum n^z_{xy} z$, where (pretend L_0, L_1, L_2 pairwise $\pitchfork$)

$n^z_{xy} = \#\text{maps} \Big\{ \xrightarrow{u} X \,\Big|\, \bar{\partial}_J u = 0$

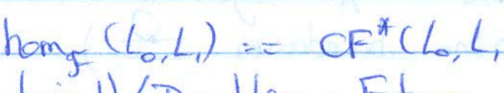
J is some almost-$\mathbb{C}$-str depending on Σ, recovering J_{L_0,L_1}, J_{L_1,L_2} and J_{L_0,L_2}. In some sense, it should coincide near y with the one for $CF^\circ(L_0,L_1)$, etc. In some sense, it should interpolate.

Point: μ^2 is a chain map (∂ 1-dim'l moduli space argument), so it descends to $[\mu^2]$ in homology.

Fukaya category $\mathcal{F}(X)$:

Objects: $L^n \subseteq X^{2n}$ exact (+ extra data for char($\mathbb{k}$) & gradings)
Morphisms: $\hom_{\mathcal{F}}(L_0, L_1) := CF^*(L_0, L_1, J, H)$
or (cohomological)/Donaldson - Fukaya category: $\mathrm{Hom}_{H^\circ(\mathcal{F})}(L_0,L_1) = HF^\circ(L_0,L_1)$, $(\mu^2$

There is this μ^2, and in general μ^k, taking a sequence of composable homs with k inputs, and returns intersection pt between end-pts of this sequence. It counts



Next eqn: μ^2 is not associative, but $\mu^2(x, \mu^2(y,z)) - \mu^2(\mu^2(x,y),z) = \mu^3(\mu^1 \otimes id \otimes id + \dots) + \mu^1 \circ \mu^3$

All the equations for the higher μ^k's make this an A_∞ - category.

Jingyu Zhao - The Fukaya category of the A_n Milnor fiber:

Recall: $\bar{\Psi} : \mathbb{C}^{n+1} \to \mathbb{C} : (z_0, \dots, z_n) \mapsto z_0^{m+1} + z_1^2 + \dots + z_n^2$.

$\Rightarrow \bar{\Psi}$ has isolated singularity, but it is degenerate. So, replace z_0^{m+1} by $p(z_0)$ of degree $m+1$: $w_0 z_0^{m+1} + w_1 z_0^m + \dots + w_{m+1}$. $\rightsquigarrow \tilde{\bar{\Psi}}$

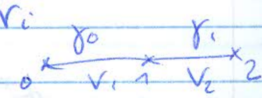
A_m Let $X_m^n := \tilde{\bar{\Psi}}^{-1}(w)$, where w is a regular value of $\tilde{\bar{\Psi}}$.

$= \{ p(z_0) + z_1^2 + \dots + z_n^2 = w \}$

If $p-w$ does not have multiple roots, X_m^n

fiber T^*S^n $\downarrow z_0$ $\Delta = (p-w)^{-1}(0)$

Let us look at the Donaldson-Fukaya category of X_m^n, or rather the subcategory with
- objects: matching spheres V_i
- morphisms: $HF^*(V_i, V_j)$.



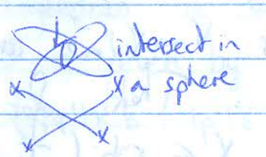
Thm [Khovanov-Seidel]:

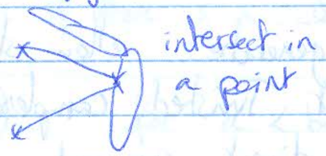
(1) If $\gamma \sim \gamma'$ isotopic (these are matching paths), $M_\gamma \cong M_{\gamma'}$ are ham isot.

(2) If γ, γ' "intersect minimally" (ie ⋔ and no bigon), then

$\operatorname{rk} HF^*(M_\gamma, M_{\gamma'}; \mathbb{Z}_2) = 2 I(\gamma, \gamma')$,

where $I(\gamma, \gamma') = \sum_{z \in \gamma \cap \gamma' \setminus \Delta} |\gamma \cap \gamma'| + \frac{1}{2} \sum_{z \in \gamma \cap \gamma' \cap \Delta} |\gamma \cap \gamma'|$





For X_m^n, $\exists$ a $\mathbb{Z}$-grading on $HF^*(L_i, L_j)$, because $c_1(X_m^n) = 0$: since 0-locus of polynomial, we can choose a holomorphic volume form η st $\eta \wedge d\tilde{\bar{\Phi}} = dz_0 \wedge \dots \wedge dz_n$.

We know that $HF^*(V_i, V_i) \overset{PSS}{\cong} H^*(S^n)$, and that a non-degenerate pairing

$$HF^d(V_i, V_{i+1}) \otimes HF^{n-d}(V_{i+1}, V_i) \to H^n(V_i, V_i) \simeq \mathbb{Z},$$

so we know that the gradings are

$$\begin{tikzcd} V_1 \arrow[r, bend right, "0"'] & V_2 \arrow[l, bend right, "n"'] \arrow[r, bend right, "0"'] & \cdots \arrow[l, bend right, "n"'] \arrow[r, bend right, "0"'] & V_m \arrow[l, bend right, "n"'] \end{tikzcd}$$

(take 1^{st} 0; then we can adjust gradings on $V_2, \dots, V_m$ so that the lower arrows are 0. Then, Poincaré duality gives the upper ones).

Algebraically, consider a graded <u>quiver</u>, and consider the path algebra of this quiver

$\overset{\cdot}{(i)} \leftrightarrow$ id in $H^*(S^n)$

path $\begin{tikzcd} \overset{\cdot}{i} \arrow[r, bend right] & \cdot i{+}1 \arrow[l, bend right] \end{tikzcd} \leftrightarrow$ class in $H^*(S^n)$

$\begin{tikzcd} i \arrow[r, bend right, "0"'] & i{+}1 \end{tikzcd} \to HF^0(V_i, V_j)$

<u>Upshot</u>: $\exists\, A_m^n$ quotient of the path algebra by $(i, i+1, i+2) = 0$
$(i, i-1, i-2) = 0$
$(i, i+1, i) + (i+1, i, i+1) = 0$

$\rightsquigarrow \mathcal{F}_m^n \cong A_m^n$

($\rightsquigarrow$ or rather, $\bigoplus_{L_i, L_j} \mathrm{Hom}_{\mathcal{F}_m^n}(L_i, L_j) = \bigoplus_{i,j} HF^*(V_i, V_j)$.

Now, what is the full Fukaya category?

Theorem: $\{V_i\}_{i=1}^m$ split generate the Fukaya category of X_m^n. ⊛

Split-generate: allow shifts $V \rightsquigarrow V[n]$ and direct sums of these, and cones. We get Twisted complexes. Then, for any L, L is quasi-isomorphic to a direct summand of $X \in \mathrm{Obj}(\mathrm{Tw}(\{V_i\}))$ in $\mathrm{Tw}(\{V_i\})$

Theorem: $\mathrm{Fuk}(X_m^n)$ for $n \geq 2$ is formal, ie $\mathrm{Fuk}(X_m^n)$ is quasi-isomorphic to $H(\mathrm{Fuk}(X_m^n))$, as A_∞-categories. <u>Not</u> true for $n = 1$.

⊛ Seidel: $\phi = \tau_{V_1} \cdots \tau_{V_m} \rightsquigarrow \phi^{2m+2}(L) = L[\sigma]$ (depends on m, n). Also, $\exists N$: $\hom^N(L, L) = 0$.
$\rightsquigarrow \phi^{(2m+2)N} = L[\sigma N]$, Get $\left(\text{twisted cplx of } V_1, \dots, V_m\right) \to L$, $L[\sigma N] \xrightarrow{\cdot 0}$ $\Rightarrow L$ is summand of this twisted complex.

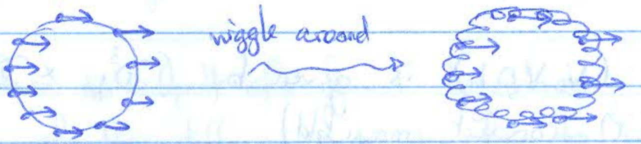
Alexander Kupers - Loose Legendrians and flexible Weinstein manifolds.

1) An h-principle is $\left\{\begin{matrix}\text{geometric}\\ \text{objects}\end{matrix}\right\} \underset{\substack{\text{hom. equiv}\\ \text{(weak)}}}{\xrightarrow{\simeq}} \left\{\begin{matrix}\text{"formal"}\\ \text{geometric}\\ \text{object}\end{matrix}\right\}$. A classical example is the Smale-Hirsch immersion theorem.

$$\left\{\begin{matrix}\text{immersions}\\ S^n \to \mathbb{R}^k,\ k>n\end{matrix}\right\} \xrightarrow{\simeq} \left\{\begin{matrix}\text{maps } f: S^n \to \mathbb{R}^k + \text{some injective}\\ \text{map } TS^n \to T\mathbb{R}^k \text{ that covers } f\end{matrix}\right\}$$

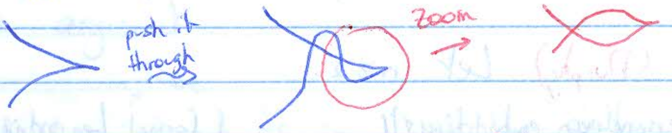
$$g \longmapsto (g, dg)$$

ex: we have a circle, and a map on tangent bundles that sends everything to a horizontal line (so, does not agree with tangent map).

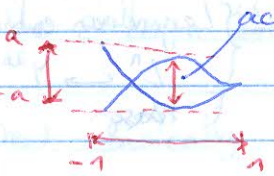


2) What is a loose Legendrian?

Stabilization:



Normalize it so it has specified dimensions: , slope $\in [-1,1]$



λ_0 "

Definition: $L^n \subseteq Y^{2n+1}$ Legendrian is loose if there is a Darboux chart $\mathcal{U}$ such that $(\mathcal{U}, \mathcal{U}\cap L)$ is given by

$$\left(R_{abc} ,\ \lambda_0 \times \{|q'| \leq b,\ |p'| \leq c\}\right) \quad \text{for } a < bc.$$

$$\downarrow$$

$$= \underbrace{[-1,1]^2 \times [-a,a]}_{\text{smallest cube containing the stabilization}} \times \underbrace{\{|q'| \leq b, |p'| \leq c\}}_{\subseteq T^*\mathbb{R}^{n-2}}$$

↑ slope is $q_1 \in [-1,1)$

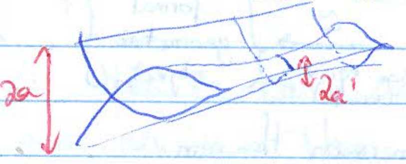
So:

b should be much bigger than a.

Rem: one can always find such a thing ~~to~~ without the assumption $a < bc$, by an h-principle of Gromov.

Properties: • You can assume $c = 1$, by a rescaling

• You can make a arbitrarily small:



For this we need time to make it small, ie b should be big. Need slope < 1.

Using Reidemeister moves, we can have that: 

So, if we have enough parallel copies of , we are also loose.

Property: if $\exists V$ st $(V, V\cap L)$ is given by $(\mathbb{D}^3_{std} \times \mathbb{D}^*_\varepsilon M, \lambda_0 \times M)$, then L is loose (M compact manifold).

Before, to make thing small, we had to cut off our scaling fct near the ends. This is not a problem if M is compact.

Theorem (Murphy): Let $n \geq 2$.

$$\left\{\begin{array}{l}\text{Legendrian embeddings}\\ \quad L^n \hookrightarrow Y^{2n+1}\\ \text{loose}\end{array}\right\}\Bigg|_{\text{loose}} \xrightarrow[\pi_0\text{-iso}]{} \left\{\begin{array}{l}\text{formal Legendrian}\\ \text{embeddings } L^n \to Y^{2n+1}\end{array}\right\}$$

$\shortparallel$ embeddings $f : L \hookrightarrow Y$ covered by bundle maps Ψ_t st $\Psi_0 = df$, $\Psi \subseteq \xi$ as Legr subspace.

So, up to deformation, loose Legendrians are classified by these formal things.

Corollary (Cieliebak-Eliashberg) let $n \geq 2$.

$$\left\{\begin{array}{l}\text{Weinstein structures}\\ \text{on } (W^{2n+2}, [\eta])\end{array}\right\}\Bigg|_{\text{flexible}} \xrightarrow[\pi_0\text{-iso}]{} \left\{\begin{array}{l}\text{"formal" Weinstein structure}\\ \quad\text{on } (W^{2n+2}, [\eta])\end{array}\right\}$$

$[\eta]$: homotopy class of non deg. 2-forms; the sympl str of W has to lie in it

$\shortparallel$ generalized Morse function with critical points of index $\leq n+1$.

A Weinstein structure is flexible if the index $n+1$ critical points have attaching spheres that are loose.

Now, we'll see why looseness is important in Emmy's theorem, and then talk a bit about Cieliebak-Eliashberg's theorem.

3) h-principle for Loose Legendrians:

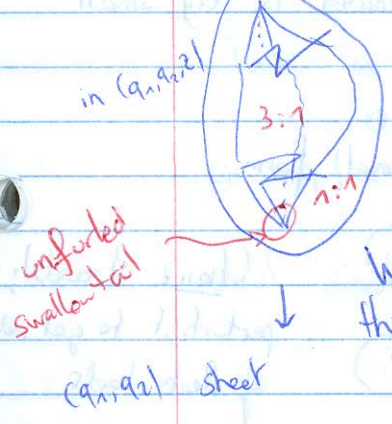
To prove that it is a π_0-surj, we need to prove that every formal Legendrian (f, ψ_t) should be connected to actual loose Legendrians by a path of formal Legendrians. For π_0-inj, we need to prove that if two loose Legendrians can be connected by a path of formal Legendrians, then this path can be deformed to a path of loose Legendrians.

1. Given a (path of) formal Legendrian, we make it a ε-Leg; ie the image of the diff is made ε-close to something which is an actual Leg'n, ie Lagr subspace of ξ. We can do it by some wiggling argument: convex hull of Lagr subspaces is entire grassmanian, so it works by some convex integration argument.

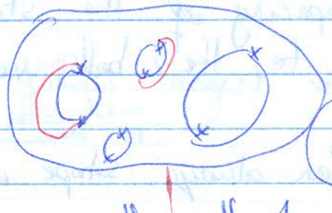
2. Use this to find (moving in time) charts in which the ε-Leg'n ~~are~~ is graphical.

3. Make ε-leg'n wrinkled Legendrian (see later). Basically, these are actual Legendrians, except on some wrinkles.

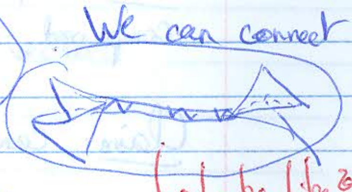
4. Use looseness to cancel out all wrinkles.



So we have wrinkles:



We can connect



singularities 2 by 2:

↳ Looks like zigzag

This doesn't really work in paths though.

We can use our zigzag, pressed down like

, and ok this guys to cancel the singularities we had.

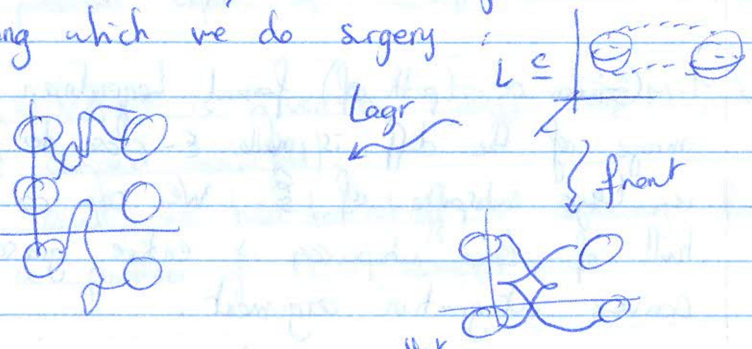
Kevin Sackel: Legendrian contact homology in the boundary of a subcritical Weinstein 4-manifold:

Ref: Ekholm, Ng

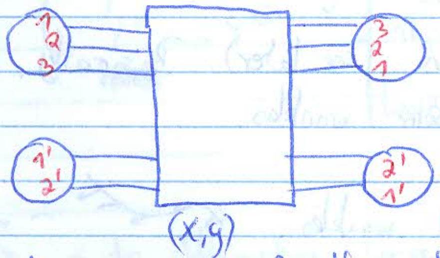
Subcritical Weinstein 4-manifold : B^4 with k 1-handles attached ; cannot have higher handles. Then $\partial W = Y_k = \#^k(S^1 \times S^2)$; we consider the "standard" contact structure on Y_k with the usual Stein filling (ie, W).

1) Combinatorial description:

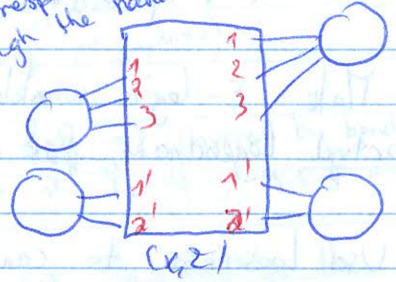
We need a normal form for a Legendrian in Y_k. Look at a copy of $\mathbb{R}^3$ in ∂B^4; then Y is just this $\mathbb{R}^3$ with some ball removed, along which we do surgery :



Definition: normal form



(numbering means that strands correspond to each other through the handle)

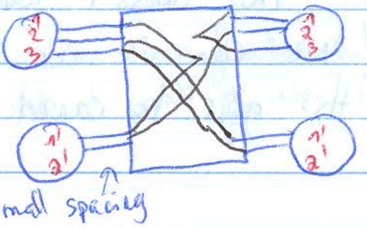


(not horiz strands)

Want: spacing of the strands going into the handles is very small compared to the balls we have removed.

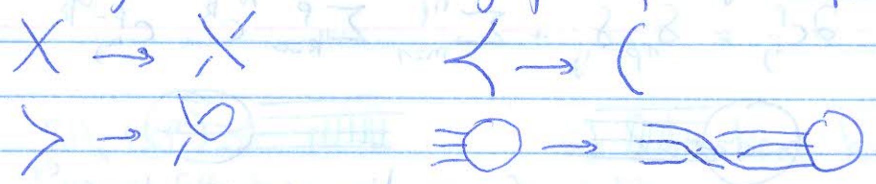
Claim: Can always isotope a Legendrian link to normal form.

Gompf standard form in (x,z) front



(horiz strands), we'll perturb it to get rid of Reeb chords

To get a normal Lagr xy-form from Gompf ~~std~~ standard form:



We resolve pictures, not Legendrians

Claim: resolving Gompf form gives a Legendrian isotopic Lagrangian projection of a legitimate Legendrian link in normal form.

Rem: the strands get to the balls we've removed in a small region; this corresponds to some path along the 1-handle; so these strands live in a neighbourhood $J^1(\mathbb{R},\mathbb{R})$ of this path.

The DGA will be defined for the xy-position of an oriented Legendrian in normal form.

* Algebra: tensor algebra over $\mathbb{Z}/2\mathbb{Z}[t_1^{\pm},\dots,t_s^{\pm}]$

$$\begin{pmatrix} \text{My leg } \Lambda = \Lambda_1 \cup \dots \cup \Lambda_s \\ Y_k = \#^k(S^1\times S^2) \end{pmatrix}$$

freely generated by
- diagram chords $a_1,\dots,a_m$
- $c_{ij;\ell}^0$ where ℓ = handle number, n_ℓ = # strands through ℓ^{th} handle, $1\le i<j\le n_\ell$

chords in the handles

- $c_{ij;\ell}^p$ $1\le i,j\le n_\ell$, $p\ge 1$. (we'll see later what they are)

* Grading: $|t_j| = -2\,\mathrm{rot}(\Lambda_j)$

Pick $*_j \in \Lambda_j$ base point. Leg'n are oriented, so $\overset{*_j}{\bullet}\!\!\longrightarrow v_j$

Let ξ_j a path in S^1 from v_1 to v_j. Let $\xi_{ij} = \xi_i^{-1}\circ\xi_j$

For intersection a_k (picture: crossing with a_k^+ and a_k^-) (a_k^+ on upper strand; a_k^- on lower one)

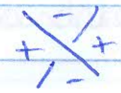
Take a path σ_i in S^1; follows tangent vectors

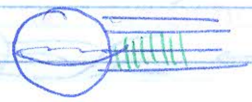
$$a_i^+ \longrightarrow *_{i^+} \xrightarrow{\xi_{i^+i^-}} *_{i^-} \xrightarrow{\text{against orientation}} a_i^-$$

so $|a_k| = \lfloor 2\,\mathrm{rot}(\sigma_k)\rfloor$

And similarly, we can define $|c_{ij;\ell}^p| = 2p-1+m(S_{i;\ell})-m(S_{j;\ell})$

* Differential: $\partial t_j^{\pm} = 0$: trivial on coefficient ring.
For fixed ℓ, $\partial c_{ij}^{p} = \delta_{1p}\delta_{ij} + \sum_{m=1}^{n_\ell} \sum_{k=0}^{p} c_{im}^{k} c_{mj}^{p-k}$ $(c_{ij}^{0} = 0 \text{ if } i \geq j)$





↳ negative corner at c_{ij}^{0}

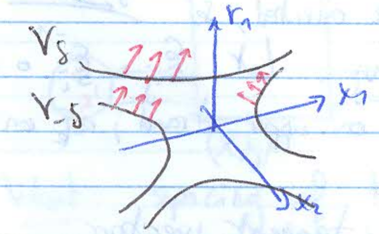
$$\partial a = \sum_{r \geq 0} \sum_{b_1,\dots,b_r} \sum_{\Delta} t_1^{-n_1(\Delta)} \dots t_s^{-n_s(\Delta)} b_1 \dots b_r,$$

where $n_k(\Delta)$ = signed count of # of times passing through $*_k$.
Δ are immersed, with boundary along Λ, with 1 positive puncture at a and negative punctures at $b_1, \dots, b_r$ in order, where b_k is $c_{ij;\ell}^{\circ}$ or a diagram chord.

Claim: ∂ has degree -1, and $\partial^2 = 0$.

Claim: DGA invariant up to "stable tame isomorphism" (for infinitely generated algebras) under Legendrian Reidemeister.

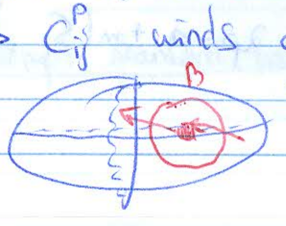
Let us explicitely describe an handle: $f(\overset{\mathbb{C}^2}{\overset{\cup}{z_1,z_2}}) = \frac{1}{2}(x_1^2+y_1^2) + 2x_2^2 - y_2^2$
$H_\delta = \{(z_1,z_2) \mid |f(z_1,z_2)| \leq \delta^2\}$ handle
$V_{\pm\delta} = \{f = \pm\delta^2\}$
We have ω_{std} on the handle, with Liouville vf $Z = \frac{1}{2}(x_1\partial_{x_1} + y_1\partial_{y_1}) + (2x_2\partial_{x_2} - y_2\partial_{y_2})$, giving the contact structure on the boundary.



Look at V_δ, I get precisely one geometric Reeb orbit.
$(x_2 = y_2 = 0)$; rotation in x_1,y_1-plane.

Standard Legendrian strand in V_δ : $x_2 = y_1 = 0$.
In $\mathbb{R}^3$, remove these contact balls. We glue it on $V_{-\delta}$, which we flow by the Liouville vf. to V_δ
↳ C_{ij}^{p} winds around p times.

$E(a) = \{|z_1|^2 + a^{-1}|z_2|^2 = 1\}$



2 Reeb orbits $E(a) \cap \mathbb{C} \times \{0\}$, $E(a) \cap \{0\} \times \mathbb{C}$. Then Reeb flow is a slope a rotation on a torus. Action $\geq K\varepsilon^{-1/p}$
$|a - m/n| \geq n^{-1/p}$.

Robert Castellano - Symplectic cohomology II:

1) $SH_*(\mathbb{D}^2)=0$
2) Action filtration
3) Useful tools
4) Viterbo functoriality
5) Subcritical handle attachment
6) Misc
7) Lagrangian obstructions

1) $SH_*(\mathbb{D}^2)=0$:

linear with slope τ at ∞

Recall $SH_*(\mathbb{D}^2) = \varinjlim HF(H_\tau, \mathbb{D}^2)$

We will construct $\tau_k \nearrow \infty$ st $SH_*(H_{\tau_k}, \mathbb{D}^2) \xrightarrow{0} SH_*(H_{\tau_{k+1}}, \mathbb{D}^2)$ for degree reasons.

Take $H_\tau(z) = \tau \cdot \frac{1}{2}|z|^2 = \tau \cdot \frac{1}{2} r^2$ (quadratic works too)

↪ $X_{H_\tau}(z) = \tau i z$

↪ flow is $\varphi_t(z) = e^{\tau i t} z$

↪ orbits:
- non-degenerate fixed point at 0
- 1-periodic orbit at $\frac{2\pi}{\tau}$.

So, pick $\tau_k \in (2\pi k, 2\pi(k+1))$ so that there is no orbit of the 2nd type. So, H_{τ_k} has 1 fixed point.

Computation: degree of 0 at a fixed pt of H_{τ_k} is $-(2k+1)$.

$$\Rightarrow HF_*(H_{\tau_k}, \mathbb{D}^2) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } * = -(2k+1) \\ 0 & \text{else} \end{cases}$$

and $HF_*(H_{\tau_k}, \mathbb{D}^2) \xrightarrow{0} HF_*(H_{\tau_{k+1}}, \mathbb{D}^2)$.

2) Action filtrations:

Recall/convention: differential decreases degree and action.

↪ $CF(H, (-\infty, a))$ = Floer complex generated by orbits of action $< a$ is well defined, if differential decreases action.

⇝ $HF(H, (-\infty, a)) = H_*(CF(H, (-\infty, a)))$

⇝ $SH_*(W, (-\infty, a)) = \varinjlim HF(H_\tau, (-\infty, a))$

Similarly, $CF(H,[a,b)) = CF(H,(-\infty,b)) \big/ CF(H,(-\infty,a))$
+ HF and SH.

3) Useful tools:

1. Action filtration exact sequence:

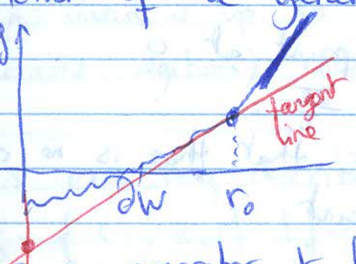
$$0 \to CF_*(H,[b,c)) \to CF(H,[a,c)) \to CF(H,[a,b)) \to 0 \qquad a<b<c$$

⇝ ~~Same~~ LES for HF by taking homology
⇝ ~~Same~~ LES for SH by taking direct limit.

2. Isotopy invariance: if $\{H_\lambda\}_{\lambda\in[0,1]}$ is a family of hamiltonians and a_λ, b_λ are numbers st none of a_λ or b_λ are equal to the action of a generator for H_λ, then

$$HF(H_0,[a_0,b_0)) \cong HF(H_1,[a_1,b_1))$$

3. Action of a generator in infinite cone



Hamiltonian in conical end is $H(\underset{\text{ham time}}{t}, \underset{\partial W\ \text{coord}}{m}, \underset{\text{cone coord}}{r}) = h(r)$; $h(r) = \overset{a>0}{ar} + b$ for $r>>0$

Claim: For a generator at level r_0, the action of this generator is the y-intersection of the tangent line.

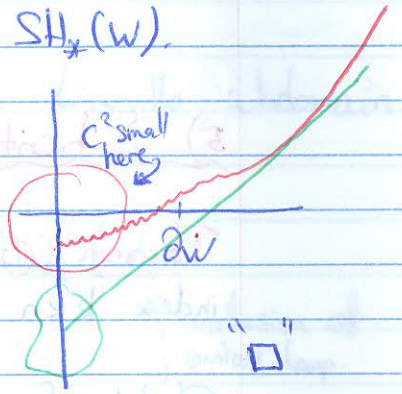
Indeed, Action $= \int_0^1 (-\gamma^*\lambda + H(\gamma(t)))\,dt$
$= -r_0 h'(r_0) + h(r_0)$ = y-intersection.

4. For a fixed action window $[a,b)$, increasing the slope of the Hamiltonian does not do anything after some point.

4) Viterbo functionality:

Theorem: $\exists$ natural map $H_{*+n}(W, \partial W) \to SH_*(W)$.

Idea of proof: take H C^2-small inside W:

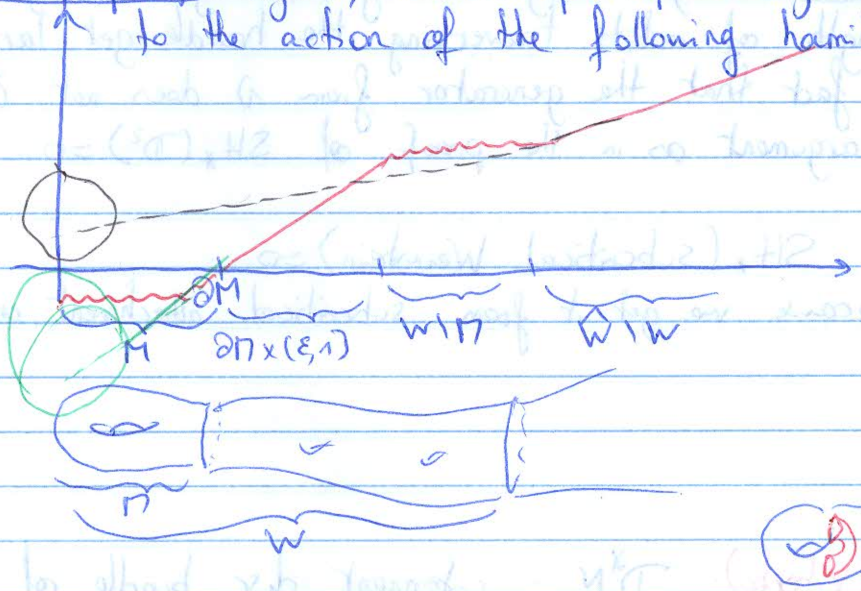


Inside W we get generators of Morse complex. We split up the generators of two types by action: those from ∂W, and the others.

"□"

Theorem [Viterbo]: Let $i : M \hookrightarrow W$ be a codimension 0 embedding st $i^*\lambda_W = \lambda_M + df$ (i.e., a "Liouville embedding"), Then, $\exists$ map $SH_*(W) \to SH_*(M)$.

Idea of proof: again, we will split up the generators, according to the action of the following hamiltonian:



We want to count generators only in M:

This can be done via an "integrated maximum principle". Floer cylinders are required to satisfy what Viterbo calls "Condition (A)".

"□"

Rem: So, to summarize, we ~~want~~ ~~to~~ have a subcomplex of $CF(W)$, and we want to check to it corresponds to $CF(M)$, including the differential.

Theorem:

$$\begin{tikzcd} SH_*(W) \arrow[r] & SH_*(M) \\ H_{*+n}(W,\partial W) \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & H_{*+n}(M,\partial M) \arrow[u] \end{tikzcd}$$

5) Subcritical handle attachment:

Theorem: (Cieliebak) Let W be obtained from M by attaching a handle of index $k<n$. Then $SH_*(W) \xrightarrow{\cong} SH_*(M)$.

Sketch of proof:

1) Make a hamiltonian on W by extending from M in such a way that there is one critical point of index k in the handle.
2) Need to make sure that no 1-periodic orbit traverses the handle:

Computation: by making the region along which you attach smaller, the length of orbits traversing the handle get larger.

3) The fact that the generator from 1) does not contribute is by an index argument as in the proof of $SH_*(D^2)=0$. "□"

Corollary: $SH_*(\text{subcritical Weinstein})=0$.
That's because we get it from subcritical attachment on the ball.

6) Mix:

Theorem (Viterbo): D^*N = cotangent disc bundle of closed N
$\mathcal{L}N$ = free loop space of N.

Then, $SH_*(D^*N) \cong H_{-*}(\mathcal{L}N)$.

Theorem (Oancea): Künneth formula for SH_*:

$SH_*(M\times N) \cong SH_*(M) \otimes SH_*(N)$

↑ ⚠ manifold with corner

Corollary: a subcritical Weinstein manifold does not contain any exact Lagrangian.

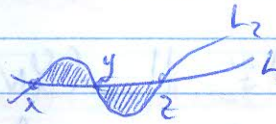
Proof: if L exact, then $D^*L \hookrightarrow W$ as a Liouville subdomain. Then we have the diagram

$$\begin{tikzcd}
0 = SH_*(W) \arrow[r] & SH_*(D^*N) \cong H_{-*}(\mathcal{L}N) \\
H_{*+n}(W,\partial W) \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & H_{*+n}(D^*L, \partial D^*L) \cong H_{*+n}(L) \arrow[u, "\text{inclusion of constant loop.}"']
\end{tikzcd}$$

The map on the right is non 0, while the one one the left is. □

Roberta Guadagni - Wrapped Floer cohomology

Recall: $HF(L_1, L_2)$ is the homology of $CF(L_1, L_2)$ generated by $L_1 \cap L_2$, and boundary map counts pseudoholomorphic curves



$\partial y = x + z$

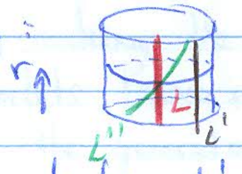
This is given by an action functional cooked up to give this.

But how to do it when $L_1 \not\pitchfork L_2$? We fix the non-transversality by translating by the flow of an hamiltonian function H: $HF(\phi_H L_1, L_2)$.

Lucky fact: in compact case, this does not depend on H.

Let $(M, \omega = d\lambda)$ be a Liouville domain, and $L_i \subseteq M$ st $\phi \neq \partial L_i \subseteq \partial M$.

Example of problem:



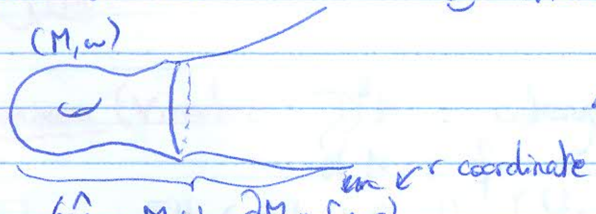
$S^1 \times \mathbb{D}^1 = D^* S^1$.

$HF(L) = ?$

If H rotates the cylinder, $L' = \phi_H L$ will be disjoint so not cool. If H rotates as a function of the height: get L''. We can take a power of this function, and get even more wrapping.

So this shows that we can not define $HF(L_1, L_2) = HF(\phi_H L_1, L_2)$, as it will give different results.

Good H: st $\lim_{r \to \infty} rH' - H = \infty$. For such H's, the answer will not depend on H. ex: $H \sim r^2$ at ∞



$\hat{M} = M \cup_{\partial M} \partial M \times [1, \infty)$

$\rightsquigarrow L \cup \partial L \times [1, \infty) = \hat{L} \subseteq \hat{M}$. So we want L to be invariant under Liouville flow near ∂M.

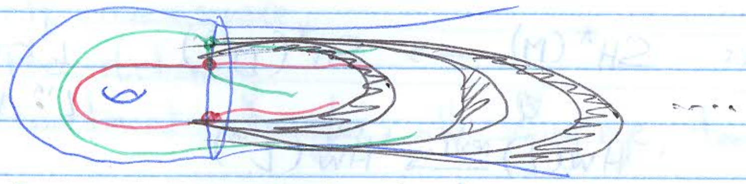
Definition 1: $HW(L_1, L_2) = HF(\phi_H L_1, L_2)$ for $H \sim r^2$ (does not depend on H)

Definition 2: if $H \sim r$, $HW(L_1, L_2) = \lim_{k \to \infty} HF(\phi_{kH} L_1, L_2)$.

In this definition, we wrap/twist more and more.

Magic: definition 1 = definition 2.

As Morgan said, we need to have some kind of maximum principle in order to make sure that we do not have holomorphic strips (between two points) going at ∞ in $\hat{M}$:

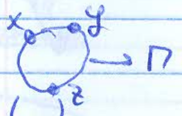


We have a product structure

$$HW(L_0, L_1) \otimes HW(L_1, L_2) \to HW(L_0, L_2)$$

$$x \otimes y \longmapsto \sum n^{z,\underline{w}}_{xy}\, z$$

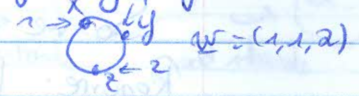
↳ counts holo discs



Actually we have $\overset{HF}{\cancel{HW}}(\phi_H L_0, L_1) \otimes \overset{HF}{\cancel{HW}}(\phi_H L_1, L_2) \to HF(\phi_{2H} L_0, L_2)$.

That's ok with def 1 (since we said it does not depend on H)

⊛ There is a way of doing it with weights... $n^{z,\underline{w}}_{xy}$ for def 2 (direct limit)



Similarly, there is a multiplication

$$SH(M) \otimes HW(L_1, L_2) \to HW(L_1, L_2).$$

$$(a, x) \longmapsto \sum_{\underline{w}, z} n^{z,\underline{w}}_{x,a}\, z$$

↳ counts

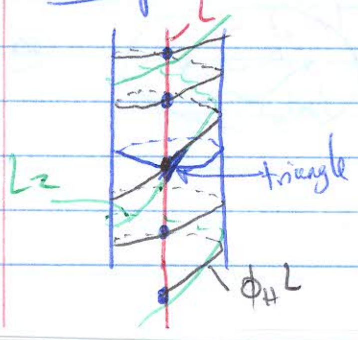
+ weights

⊛ Why would we do it with def 2? Mark: in SH talk, we needed the direct limit thing to define certain things.

HW is a unital SH-module, so $HW = 0 \Leftrightarrow 1_{HW} = 0_{HW}$.

So, $SH = 0 \Rightarrow HW = 0$.

Example: $T^*S^1 \supseteq L$ cotangent fiber



$H = h^2 \quad dH = 2h\,dh$

All the generators have the same degree, and there is no strip.

So $CW(L): \ldots 0 \to \ldots \to 0 \to \mathbb{K}^N \to 0 \to \ldots$

Let's look at the product structure

Looking at the product structure, we get $0 \to \Bbbk[x, x^{-1}] \to 0$. We notice that it is actually $H_*(\Omega S^1)$.

<u>Theorem</u>: ($*$ up to sign issues) $HW^*(T_q^* X) \cong H_{-*}(\Omega_q X)$.

↳ algebra str induced by Pontryagin product.

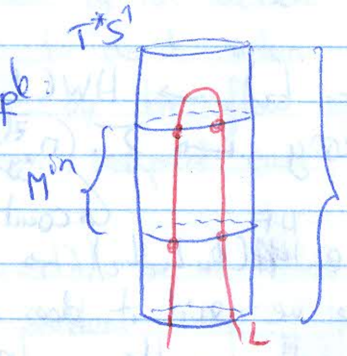
Viterbo restriction:

$$\begin{tikzcd} SH^*(M) \arrow[r] & SH^*(M^{in}) \\ HW(L) \arrow[r, "\text{restr.}"] & HW(L^{in}) \end{tikzcd}$$

$L \subseteq M$

$L^{in} = M^{in} \cap L$

Bad example:



$HW(L) = 0$

but $HW(L^{in}) \neq 0$.

That's a problem because we want r to be a map of unital rings.

So we can define this restriction functor only for <u>some</u> Lagrangians.

<u>Require</u>: $[\theta_L] = 0 \in H^1(L, \partial L \cup \partial M^{in})$.

Then, there is a restriction map.

Roger Casals - A dictionary for Lefschetz bifibrations and Legendrian fronts I:

1. A_k reminiscence
2. The dictionary
3. Assortment: triangles, all seeing eye, two andalusian dogs, celtic triad.

1) A_k reminiscence:

$$A_k \xrightarrow{\pi} \mathbb{C}$$

$\shortparallel$

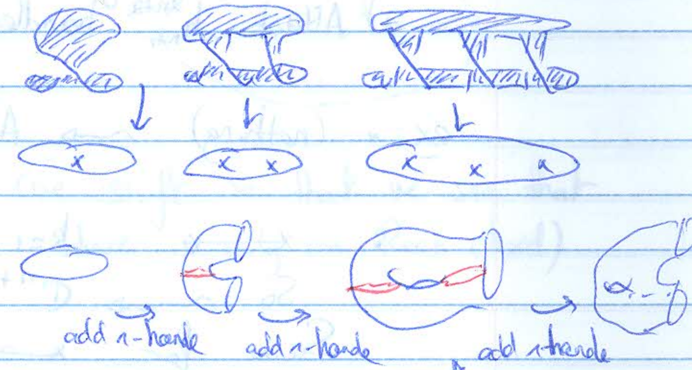
$\{z^{k+1}+w^2=5\}$

$z\in\mathbb{C},\ w\in\mathbb{C}^n$

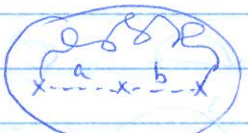
$\cong$

plumbing of k copies of (T^*S^n, λ_0)

ex:



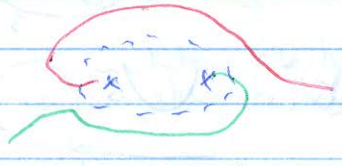
Fix a basis of matching paths:



(red on above)

We are going to look at curves that can be obtained from half-twists. (half Dehn twist)

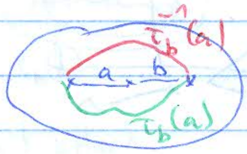
$\in \pi_0 \operatorname{Diff}(\mathbb{D}^2)$



Not: $t_b(\gamma) :=$ half Dehn-twist along b of some γ.
$\widetilde{t_b(\gamma)}$ = lift of the curve to the surface.

Fact: $\widetilde{t_b(\gamma)} = \tau_{\tilde{b}}(\tilde{\gamma})$ cf. Donaldson's Riemann Surfaces

ex:



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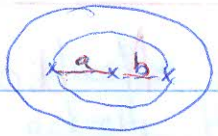
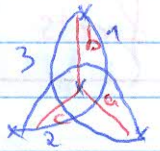
Examples: 1. Eye: 

$1 = \tau_b(a)$
$2 = \tau_b^{-1}(a)$

2. Triad: 

$1 = \tau_b(c)$
$2 = \tau_a(b)$
$3 = \tau_c(a)$

2) The dictionary:

$A_k^{2n} \times \mathbb{C} \underbrace{\rightsquigarrow}_{\text{matching paths}}^{\text{(ordered)}} W^{2n+2}$

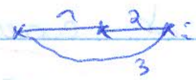
1) Lift the Lagr. in the fiber in $A_k^{2n} \times S^1$
2) Attach h_{n+1}^{2n+2} handles

ex: • (nothing) $\rightsquigarrow A_k^{2n} \times \mathbb{C}$

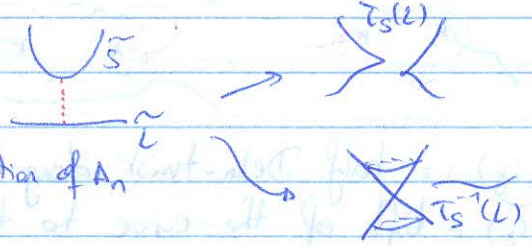
•  $k=1 \rightsquigarrow$ fiber is T^*S^n, and we attach the 0-section.
So $\rightsquigarrow \mathbb{C}^{n+1}$
Same for .

•  $\mathbb{C}^{n+1}$ and a critical handle $\rightsquigarrow T^*S^{n+1}$

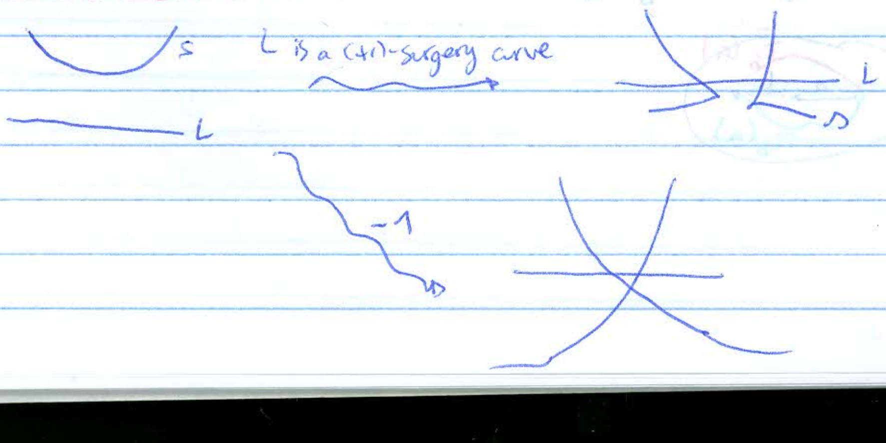
• 

or 

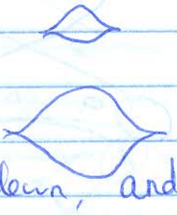
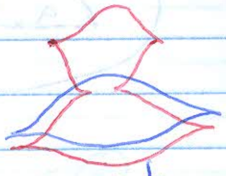
: now the drama starts.

Dehn twist front: 

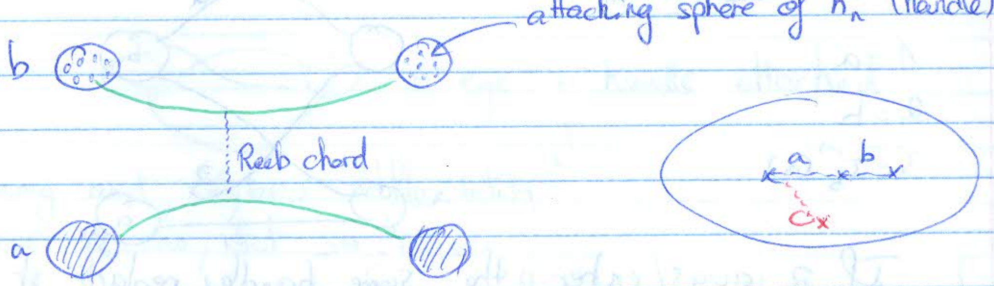
front in contactisation of A_n

Legendrian handle slide: 

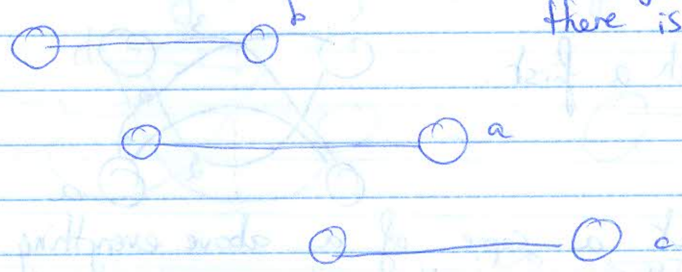
Globally: plane with twist along  $= T^*S^n$

~> in this,  is isotopic to  : we push the first one down, and apply the previous picture.

How to draw $A_k^{2n} \times \mathbb{C}$?



If we want to add c: (we shift so that we see that there is only one Reeb chord)

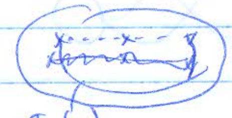
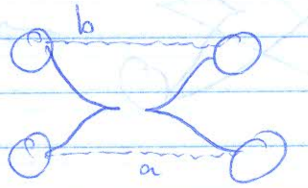


Now: 1)  =  $= \mathbb{C}^{n+1}$

2)  =  whatever else happens should be here $= T^*S^{n+1}$

|| handle slide

 = 

3)  $\tau_b(a)$ =  ; we can cancel : 

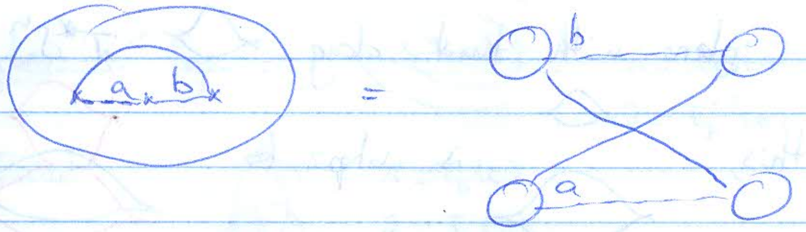
(core intersects cocore once) ~> handles cancel

(we don't attach a and b)

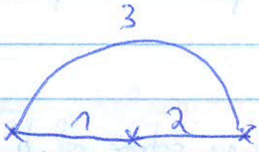
$= T^*S^n \times \mathbb{C}$

(we don't attach a & b)

4)



5)



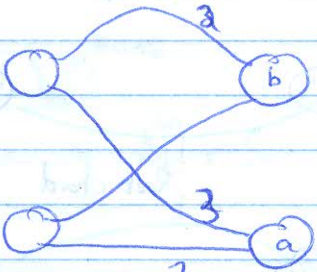
Cyclic order; we have to pick the first one; say 1 ✓

3: above everything

1 = a

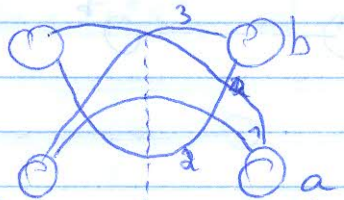
2 = b

$3 = \tau_b^{-1}(a)$



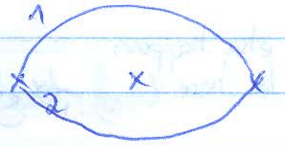
If 2 guys enter the same handle, replace it by a cusp. After some Reidemeister moves, get 

5,5) If we attach a first:



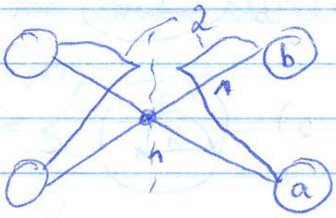
For 3: put a copie of a above everything, then a copie of b above everything, and then do a conic resolution.

6)

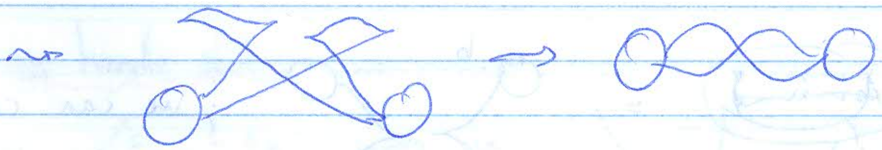


$1 = \tau_b^{-1}(a)$

$2 = \tau_b(a)$

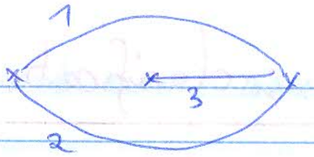


For 2: put a copie of a above, then a copie of b above, and then do cusp resolution.



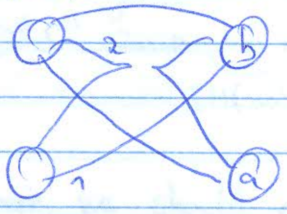
(we have 2 choices: we get the Andalusian dog - See the movie)

7) Add b:



Start with 1:

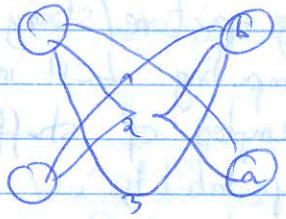
$\rightsquigarrow$



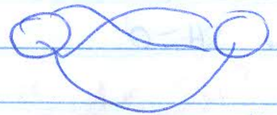
Sorry, that's the other dog $\rightarrow$

or

Start with 3



$=$



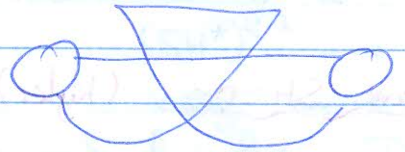
= eye + handle attached

Emmy and Kyler: "subflexibility".

Eye: not flexible, but $SH = 0$

Here: flexible, obtained from nonflexible dog attaching an handle. This is what Emmy and Kyler call subflexibility.

$\rightsquigarrow$



$\rightsquigarrow$

$\rightsquigarrow$

, which is loose.

Claim/exercise: do the other dog. It is also flexible: loose.

$=$

Review session ; clarifications , remarks, ...

1. Big picture/story
2. Computing LCH in dim >1
3. Examples of stuff
4. Computability
5. Examples / Roger's HW in slow motion
6. Why flexible manifolds have SH=0
7. τ_v^2 in smooth category
8. Lefschetz wrapping hamiltonians
9. Structures on Floer theories
10. Kirby calculus
11. SH^* computations
12. Exact triangles

6) Why flexible manifolds have SH=0: (Kyler)

Theorem: Let $(W^{\geq 6}, \lambda, \phi)$ be a flexible Weinstein domain ~~manifold~~. Then, $SH(W,\lambda)=0$.

Theorem: (Eliashberg-Murphy) h-principle on Lagrangian caps. Let $F: W^{2n} \hookrightarrow (X^{2n}, \theta)$ be a smooth codimension 0 embedding st $F^*d\theta$ is homotopic to $d\lambda$ through non-degenerate 2-forms. Then, F is smoothly isotopic to an exact symplectic embedding $(W, \varepsilon\lambda) \hookrightarrow (X,\theta)$, $\varepsilon>0$.

as in previous thm

Rem. $(W,\lambda) \hookrightarrow (Y,\alpha)$ an exact codim 0 symplectic embedding; Then $SH(Y,\alpha)=0 \Rightarrow SH(W,\lambda)=0$.

Proof of vanishing thm: it suffices to show that $SH(W\times D^*S^1)=0$, since by Künneth: $SH(W\times D^*S^1) = SH(W) \otimes \underbrace{SH(D^*S^1)}_{= H(\mathcal{L}S^1) \neq 0}$.

Consider an embedding $W \times D^*S^1 \hookrightarrow W \times \mathbb{D}^2$
identity × ()

This is definitely not exact, since  is quite far from being exact.

By E-M, we get an exact symplectic embedding

$$W \times D^*S^1 \hookrightarrow \underbrace{W \times \mathbb{D}^2}_{SH(W)\otimes \underbrace{SH(\mathbb{D}^2)}_{=0}} \quad ; \text{ not possible.}$$

□

<u>Rem</u>: somewhere, we should prove that $W \times D^*S^1$ is flexible.

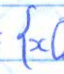
x) What is $1 \in SH$? (Sheel)

Cheap answer: there's a map $H^*(X) \longrightarrow SH^*(X)$; then the identity is the image of 1. $\quad H^*(X) \overset{\text{Poincaré}}{\cong} H^*_*(X,\partial X)$

Other answer: there is a "pair of parts" product

$$(SH^*)^{\otimes 2} \longrightarrow SH^*$$

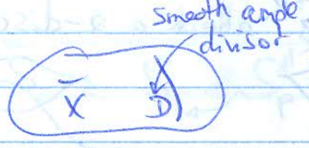
$x \otimes y \longmapsto \sum \#\{z$  $\}\cdot z$

1 is the unit for this. Define $1 = \sum_x \#\{x$  $\}\, x$.

Check that it's 1: 

but we didn't mod out by $\mathbb{R}$, so there is just the identity.

<u>Rem</u> we have a map $H_*(X,\partial X) \to SH^*(X)$; $\left(\mapsto \sum_z \#\{z\right.$  $\}\cdot z$.

<u>Rem</u>: 

$X = \bar{X} \setminus D$. There is a SS

$$H^\circ(X) + t\, H^\circ(S_D)[t] \Rightarrow SH^\circ(X)$$

↖ unit normal

$H^\circ(X)$: comes from crit pt of Morse fct

$H^\circ(S_D)[1]$: Reeb orbits in nhbd of D

$H^\circ(S_D)[t]$: iterates of these.

Diego's thesis also identifies the differentials in the SS.

3) Examples of stuff: (Emmy)

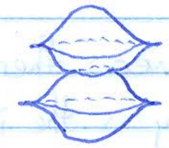
Note: for Leg'n $S^2 \subseteq \mathbb{R}^5_{std}$, <u>everything</u> is formally Leg'ian isotopic. If we think of $\mathbb{R}^5 = \partial B^6 \setminus pt$, we can attach an handle; we get some (uniquely defined, since everything's the same) Weinstein structure on $S^3 \times \mathbb{R}^3$

ex: 

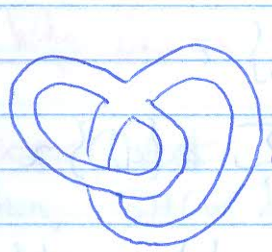
$= T^*S^3$

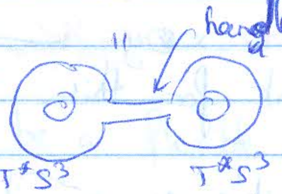
$= \partial(\{y=0\}) \subseteq B^6$

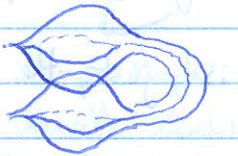
we get 0-section, as the core.

Every smooth sphere is isotopic to 

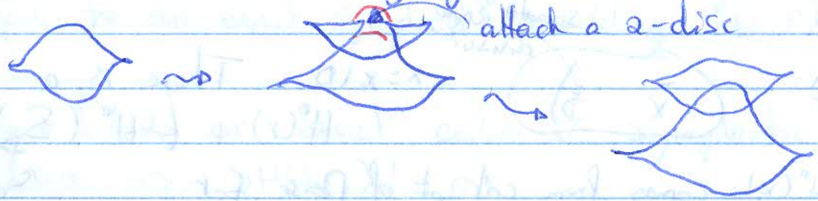
Consider the link  ≠ 





Link them together: , diffeo to $S^3 \times \mathbb{R}^3$ (because sphere)

It contains ~~ax~~ an exact Lagrangian $S^2 \times S^1$.

Start with 

$= \partial(S^2 \times I)$,

if we start with B^3 and attach a 2-handle, that's what we get. If you attach a 1-handle, get $(S^2 \times S^1) \setminus B^3$, and then we attach a maximum on top, so we get $S^2 \times S^1$.

Take 

, and push the lower one through the upper one. But not push a disc; push this:  (Bourgeois, Sabloff). The result contains one exact lagarangian, by).

If we had a S^1 worth of zigzags on the top, it becomes  .

i) Computability (Mark)

Seidel's A biased view of symplectic cohomology, section 6c.
Suppose we have a finitely generated group with a finite number of relations: $\langle g_1, \dots, g_k \mid r_1, \dots, r_m \rangle$. Is it trivial?
ex: $P = \langle g_1, g_2 \mid g_1 g_2 g_1^{-1} = 0 \rangle$ or $\langle 1 \rangle$.
Novikov: there is no computer algorithm telling us if G_p is trivial or not.
↳ gp associated to p

Now, for each group presentation, we want to create a Weinstein manifold. We need $H_1(G) = H_2(G) = 0$. Even restricted to those, Novikov proved the same thing:
Novikov: for each such P, $\exists$ an (explicitely constructed) homology sphere S_p with fundamental group G_p such that

$$S_p \underset{\text{diffeo}}{\cong} S^6 \iff G_p = \{id\}.$$

↳ This is an uncomputability result for 6-manifolds.

Let $\bar{S}_p = S_p \# S_p^{\# S_p}$. Then $\underset{\substack{\shortparallel \\ G_p * G_p * G_p}}{\pi_1(\bar{S}_p)}$ is infinite $\iff G_p \neq \{id\}$.

So there is also no way of ~~saying~~ telling if a 6-manifold has a finite fundamental group.

Let $W_p = T^* \bar{S}_p$. Then, $SH^*(W_p) = H_{-*}(\underset{\text{free loop space}}{\mathcal{L}} \bar{S}_p)$
If $G_p = \{id\} \Rightarrow \bar{S}_p = S^6 \Rightarrow H_0(\mathcal{L}\bar{S}_p) = \mathbb{Z} \Rightarrow SH^0(W_p) = \mathbb{Z}$
If $G_p \neq \{id\} \Rightarrow \pi_1(\bar{S}_p)$ has an ∞ number of conjugacy classes $\Rightarrow \operatorname{rk} H_0(\mathcal{L}\bar{S}_p)$ is infinite $\Rightarrow SH^0(W_p)$ has ∞ rank.

So, ~~there~~ it is also hard to tell if SH has finite rank.

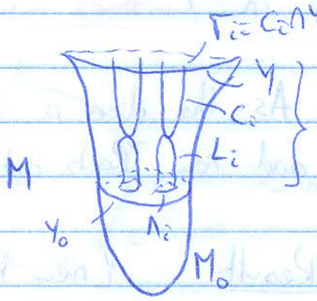
No, attach a Weinstein 2-handle along a loop representing each generator $g_1, \dots, g_k$ of P. So we get the presentation
$P' = \langle g_1, \dots, g_k \mid g_1 = 0, \dots, g_k = 0 \rangle$.
So we get $W_P = T^* S_{P'}$.

Lemma: W_P' is symplectomorphic to W_P $\Leftrightarrow$ $G_P = id$.

Zack Sylvan - The Bourgeois - Ekholm - Eliashberg surgery formula:

1) Intro:

Notation:



W = union of critical handles = Weinstein cobordism from Y to Y_0

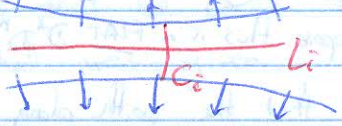
$Y = \partial M$

$Y_0 = \partial M_0$

C_i = corresponding cocore of handle attached to Λ_i:

$\Lambda = \cup_i \Lambda_i$

$C = \bigcup_i C_i$ union of cocores



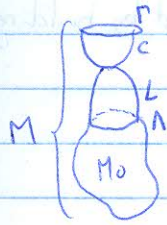
arrows = Liouville vf

SFT style invariants
- SH (Reeb orbits)
- HW – linearized LCH (Reeb chords), wrt filling of Leg'n: homology of Γ_i wrt filling by cocore
- LCH (Reeb chords) doesn't care about the filling. Of Λ.

SFT pictures:

Features of SFT: nice things it {can / can not} do relative to what hamiltonian things can do:
- more flexible pictures :)
- Foundations still in progress
- Only over $\mathbb{Q}$
- Neck stretching a lot easier in this setting
- Don't have to deal with algebraic limits, as in SH.

⚠ All of this works if we consider some action bounds.

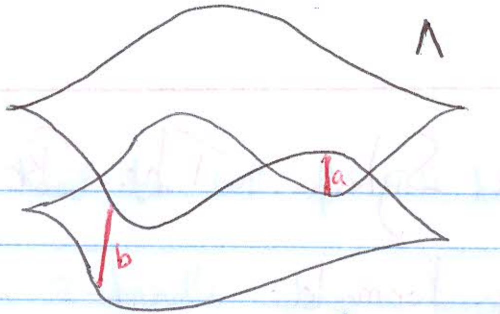


Formulas: • $HW(C) \cong LCH(\Lambda)$; there is an A_∞ chain-level map.
$\quad \curvearrowleft A_\infty \qquad \curvearrowleft$ DGA

• $SC(M) \cong \text{cone}(SC(M_0) \xrightarrow{F} LC^{H_0}(\Lambda))$, ($H_0$ = Hochschild homology of that algebra)
where F is an actual geometric map.

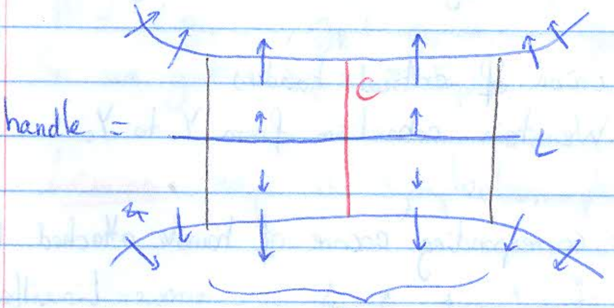
$$\begin{tikzcd}
SH(M) \arrow[rr, "\text{Viterbo transfer}"] & & SH(M_0) \arrow[dl, "F"] \\
& LH^{H_0}(\Lambda) \arrow[ul] &
\end{tikzcd}$$

is an exact triangle.



2) Dynamics:

<u>Question</u>: what are the new orbits in Y?



can suppose this is a FLAT $D^k D^n$

As the disc is flat, we know geodesics, and how Reeb orbits propagate in the handle.

<u>Result</u>: $\{$new Reeb orbits$\}$
$\downarrow$
$\{$cyclic words of Reeb orbits$\}$, emphasizing that the path along the core is no extra information.

↳ where Hochschild comes from

<u>Claim</u>: that map is a bijection.

<u>Also</u>: there is a corresponding statement for chords.

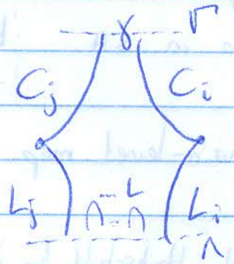
3) What does HW look like in SFT?

HW(C) is the homology of a chain complex CW(C), generated by:

- $\mathcal{X}(\Gamma)$:= Reeb chords of Γ
- e_i (Morse minimum) for each C_i, unit. The "total" unit is $\sum_i e_i$; each e_i is the projection of that onto "things that care about C_i".

<u>Differential</u>:
- $\delta e_i = 0$
- $$\delta\gamma = \sum_{\gamma'} \#\left\{ \mathbb{R}\times\Gamma_j \,\Big|\overset{\gamma}{\underset{\gamma'}{\ }}\Big|\, \mathbb{R}\times\Gamma_i \longrightarrow \mathbb{R}\times Y \right\}\gamma' + \sum_i \#\left\{ \overset{\gamma}{\underset{e_i}{\ }} e_i \longrightarrow W \right\} e_i$$

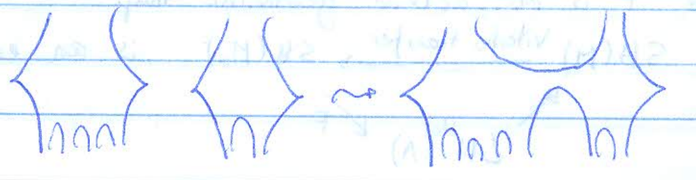
↳ these things need to be anchored, ie completed by buildings



: $\mathcal{X}(\Gamma) \longrightarrow \overset{\text{tensor algebra}}{T}(\mathcal{X}(\Lambda))$

<u>Claim</u>: this is a chain map $CW(C) \to LCC(\Lambda)$

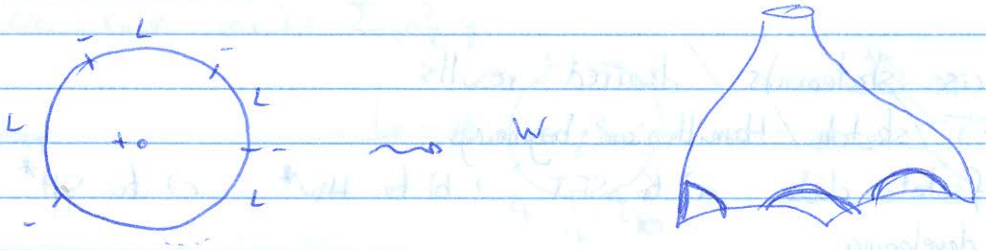
Product structure:



(product structure below is concatenation)

And similarly, we can get an A_∞-structure.

4) Morse-Bott:



This is this kind of things that we count to define the complexes in the second formula that was given.

Sheel Ganatra - SH^* and HW^* in the context of Lefschetz fibrations

1. Precise statements / desired results
2. SFT sketch / Hamiltonian beginnings
3. Lefschetz data a) to SFT b) to HW^* c) to SH^*.

⚠ Still developing...

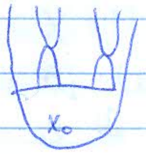
1) Precise statements / desired results:

Wanted: Compute $SH(X)$ - easy when X is flexible
- easier to compute $HW^\cdot$, ~~L~~CH in general

Precise statement v1: let $\mathbb{k}$ be an arbitrary field, X Weinstein of finite type.

$SH^\circ(X) = 0 \iff HW(L) = 0 \quad \forall L$.

$\boxed{\Rightarrow}$: from "module structure $1=0$" argument



$\boxed{\Leftarrow}$: $\left\{\begin{array}{l}\text{when} \\ \text{if}\end{array}\right.$ $\{C_i\}$ split generates, then $HW(L)=0 \ \forall L$ is equivalent to $HW(C_i) \overset{?}{=} 0$ for $\{C_i\}$ fixed.

Slightly more precise relationship: let $\mathcal{W}_c$ denote the wrapped Fukaya category of X with objects $\{C_i\}$. Then, there should be an isomorphism

$$\underset{\text{Hochschild homology}}{HH_\circ(\mathcal{W}_c)} \cong SH^\circ(X)$$

$$\text{Cor: of } \mathcal{W}_c \text{ split-generate} \xrightarrow{\quad} \ \cong \quad HH(\mathcal{W}(X)) \quad \cong$$

Actually, there is more: $HH_\circ(\mathcal{W}_c) \cong SH(X) \cong \underset{\text{Hochschild cohomology}}{HH^\circ(\mathcal{W}_c)}$

And $HW^\cdot(L)$ is a unital module over $HH^\circ(\mathcal{W})$.

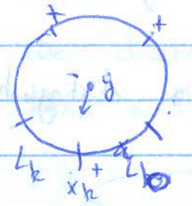
As a chain complex, $HH_\circ^\cdot(\mathcal{E})$ is (the homology of a complex) gen by $\underset{X_0,\dots,X_k}{\bigoplus} \underbrace{\hom(X_k,X_0)\otimes\hom(X_{k-1},X_k)\otimes\cdots\otimes\hom(X_0,X_1)}_{\text{cyclic chains}}$,
and the differential involves all $\mu^\cdot$

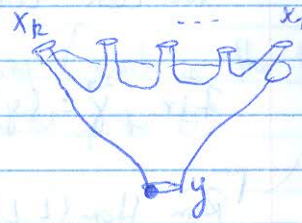
We should think of these as chains of Reeb chords, and sometimes Morse crit. pts.

For cohomology: $\prod_{X_0,\dots,X_k} \hom_{vect}\left(\hom(X_{k-1},X_k)\otimes\dots\otimes\hom(X_0,X_1),\ \hom(X_0,X_k)\right)$.

Why should these relations exist? There are geometric maps
$HH_\bullet(\mathcal{W}) \xrightarrow{[oc]} SH^\bullet(x) \xrightarrow{[co]} HH^\bullet(\mathcal{W})$, where
$OC(x_k\otimes\dots\otimes x_1) = \sum \boxed{n^y_{\vec{x}}}\, y$

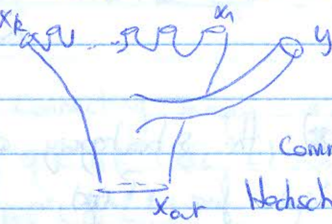
$[co] = co$ on homology

$\hookrightarrow = \#$ 

$= \#$ 

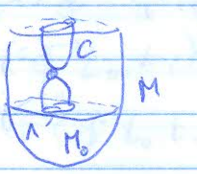
this OC should commute with Viterbo restriction.

CO $\longleftrightarrow$ 

$\longleftrightarrow$ 

; this is not expected to commute with Viterbo because Hochschild cohomology is not functorial.

Theorem (Ganatra) $[OC]$ hits 1 $\Leftrightarrow$ OC & CO are isomorphisms.

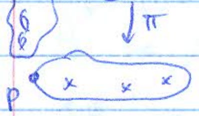
SFT version: 

[BEE]: $\overset{\text{in } M}{HW_{SFT}(C)} \underset{A_\infty}{\cong} LCH(\overset{\text{in } M_0}{\Lambda})$, (there should be C's instead of H's at some places)

and $SH(M) \cong LH^{H_0}(\overset{\text{in } M_0}{\Lambda})$

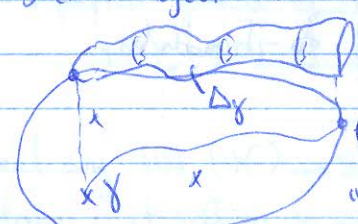
Algebraic fact: $LH^{H_0}(\Lambda) \cong HH_\bullet(\underbrace{LCH(\Lambda)}_{\text{as DGA}})$.

Let us analyse the situation in presence of a Lefschetz fibration.



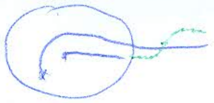
Slogan: there are simpler formulae for HW/CW & SH in this case! (in terms of Fukaya structures associated to (E,π)).

(i) Fukaya category of (E,π): (Seidel, Abouzaid-Seidel to appear)
$\mathcal{F}(x)$: objects are Lefschetz thimbles for paths γ from crit value in $\mathbb{C}$ to p.



$\hom_{\mathcal{F}(x)}(\Delta_i,\Delta_j) = CF^*(\phi_\varepsilon \Delta_i, \Delta_j)$
$\hookrightarrow$ depends on ε, x
"time λ ccw bend in $\mathbb{C}$, pulled up to E via π)
Asymmetric choice of perturbations! (clockwise)

in $\mathbb{C}$:

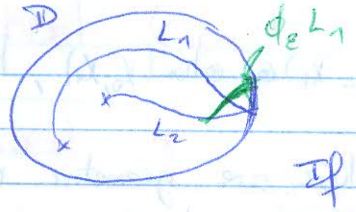


thimbles // to $\mathbb{R}^+$ near ∞

$\hom(L_2, L_1) =$ something

$\hom(L_1, L_2) = 0$

<u>ex:</u>

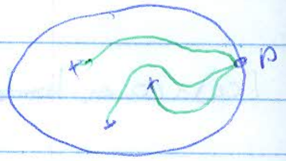


. We perturbe only near the boundary.

If we instead perturbe L_1 a little bit up, there is no more intersection ⍨.

We can compose, and get an A_∞-structure...

Fix $\vec{\gamma} = (\gamma_1, \ldots, \gamma_k)$ a distinguished basis of vanishing paths:



Rem: $\mathrm{Hom}(L, L)$:

$\rightsquigarrow$ they intersect only at the origin

$\rightsquigarrow \hom(L, L) = \Bbbk$

We get $\mathcal{F}^{\vec{\gamma}}(\pi)$ the subcategory of thimbles Δ_{γ_i}. There is a model entirely in $\Pi = \pi^{-1}(p)$ for $\mathcal{F}^{\vec\gamma}(\pi)$ (Seidel).

[Seidel]: $\mathcal{A}_{(\gamma)}$: objects $V_{\gamma_i} =: V_i$ vanishing cycles of γ_i

$$\hom(V_i, V_j) := \begin{cases} CF^*(V_i, V_j) & i < j \\ \Bbbk & i = j \\ 0 & i > j \end{cases}$$

(// what we get with these perturbations above)

$\rightsquigarrow \mathcal{A}_{(\gamma)} \overset{A_\infty}{\cong} \mathcal{F}^{\vec{\gamma}}(\pi)$

Often think of $\mathcal{A}$ as an $\overset{A_\infty}{}$algebra via $\bigoplus_{i \leq j} \hom(V_i, V_j)$ over the semi-simple ring $R = \bigoplus \Bbbk e_i$.

There is also $\mathcal{B}_{(\gamma)} \underset{\text{full}}{\subseteq} \mathcal{F}(M)$, objects $= \{V_i\}$, and there is no directedness, ie $\hom(V_i, V_j) = CF^*(V_i, V_j)$ $\forall i, j$.

Note that there is an inclusion $\mathcal{A}_{(\gamma)} \hookrightarrow \mathcal{B}_{(\gamma)}$.

<u>Claim:</u> [Seidel, Abouzaid - Seidel]: $\mathcal{A}_{(\gamma)} \hookrightarrow \mathcal{B}_{(\gamma)}$ determines the wrapped Floer homology of thimbles.

Actually, we just need a map $\mathcal{A} \to \mathcal{B}$ of $\mathcal{A}$-bimodules.

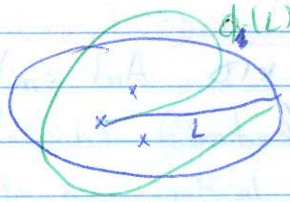
Cool. Now, let's go back to the total space.

$$\begin{tikzcd} \mathcal{A}_{(\gamma)} \arrow[d, phantom, "\shortparallel"] & \\ \mathcal{F}_{(\gamma)}(\pi) \arrow[r, "\cap", "\text{functor}"'] & \mathcal{F}(M) \end{tikzcd}$$

In the presence of (E, π), we can "discretize" wrapping / work in the base:
Denote by ϕ the "time 2π bend in the base" (pulled up to E)

$$\phi : \mathcal{F}(\pi) \longrightarrow \mathcal{F}(\pi)$$

Boundary Dehn twist



Point: $HW^0(\Delta_i, \Delta_j) = \varinjlim_{k} HF^0(\phi^{k+\varepsilon}\Delta_i, \Delta_j)$

The connecting maps $\rightsquigarrow$ $id \xrightarrow{\overset{\text{natural transf.}}{t}} \phi$

To first order, an element $q \in HF^0(\phi L, L) \overset{d(1)}{\rightsquigarrow}$ such that multiplication by q induces $HF^0(\phi L, L) \to HF^0(\phi^2 L, L)$

[Seidel]: $\{\phi^{-1} \to id\}$ equivalence formulation

Geom/alg fact: ϕ^{-1} is a Serre dualizing functor.

$$\underset{\overset{\shortparallel}{CF^0(\phi^{-1}L_0, L_1)}}{hom(\phi^{-1}L_0, L_1)} = hom(L_1, L_0)^\vee$$

Point: the wrapped category is the localization of $\mathcal{F}(\pi)$ with respect to
$\{\phi^{-1} \leftrightarrow id\} \leftrightarrow id \to \phi$.

What does this have to do with $\mathcal{A}$ and $\mathcal{B}$? For $\mathcal{A}_{(\gamma)}$, $\mathcal{A}^\vee$ as an $\mathcal{A}$-$\mathcal{A}$ bimodule is also a dualizing functor

$\mathcal{A}^\vee \xrightarrow{\delta} \mathcal{A}$ has $$\begin{tikzcd} \mathcal{A} \arrow[r] & \mathcal{B} \arrow[r] & \mathcal{B}/\mathcal{A} \simeq \mathcal{A}^\vee[-1] \arrow[ll, bend left, dashed, "\delta"] \end{tikzcd}$$

$\{\phi^{-1} \to id\}$

- Simplified formula of HH_0(localized categories) [Seidel, Abouzaid - Ganatra]

$$HH_0(\mathcal{W}) \cong \varinjlim_{k} HH_0(\underset{\overset{\shortparallel}{\mathcal{A}}}{\mathcal{F}_\gamma(\pi)}, \overset{\nwarrow \text{ a functor}}{\phi^k}) = \varinjlim_{k} HH_0(\underset{\overset{\shortparallel}{\mathcal{A}}}{\mathcal{F}_\gamma(\pi)}, \phi^{-k})^\vee$$

- Realizing $HH_0(\mathcal{W}) \to SH^0(E)$ using hamiltonians $\cap$ / localizations [Abouzaid - Ganatra]

$HF^0(\pi)$ between $H^0(E)$ & $SH^0(E)$

Claim: "it suffices to wrap once": $\exists$ diagram

$$\begin{tikzcd} HH_0(\mathcal{F}(\pi), \phi) \arrow[r] \arrow[d, "OC_\pi"] & HH_0(\mathcal{W}) \arrow[d, "OC"] \\ HF(\pi) \arrow[r, "1 \mapsto 1"] & SH^0(M) \end{tikzcd}$$

Fact: if OC_π iso, then OC iso ||| Floer theoretic invariant $\nearrow$ $1 \in HF(\pi)$

90

Mirror Symmetry: outside CY situation,

$$X \text{ Fano} \longleftrightarrow (\check{X}, W)$$ ← Lefschetz fibration $W : \check{X} \to \mathbb{C}$

D anticanonical divisor

$$\begin{tikzcd}
D^b\mathrm{Coh}(X) \arrow[r, leftrightarrow, "\mathrm{HMS}"] \arrow[d, "i^*"'] \arrow[dd, bend right=60, "\mathrm{loc}"'] & D^\pi \mathcal{F}(W) \arrow[d, bend left, "\cap"] \arrow[dd, bend left=60, "\mathrm{loc}"] \\
D^b\mathrm{Coh}(D) \arrow[r, leftrightarrow, "\mathrm{HMS}"] \arrow[u, bend left, "i^*"] & D^\pi \mathcal{F}(W^{-1}(p)) \arrow[u, "\cup"'] \\
D^b\mathrm{Coh}(X \setminus D) \arrow[r, leftrightarrow, "\mathrm{HMS}"] & D^\pi \mathcal{W}(\check{X})
\end{tikzcd}$$

// if we know X and D, we can recover $X \setminus D$. And similarly, if we know W and $W^{-1}(p)$, we can recover $\check{X}$.

Oleg Lazarev - Exotic symplectic structures on T^*S^{n+1}

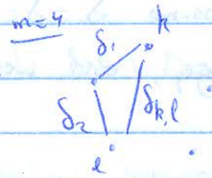
1. Construction / results
2. Spectral sequence
3. Fukaya category
4. A_m - quiver representation

1) Construction / results:

Lefschetz fibration: $T^*S^{n-1} \hookrightarrow M_m^{2n} = \{ x_1^2 + \dots + x_n^2 + x_{n+1}^{m+1} = 1 \}$ with $m+1$ critical values $\sqrt[m+1]{1} \subseteq \mathbb{C}$. $\qquad \downarrow x_{n+1}$ $\mathbb{C}$

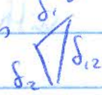
Matching path $\delta \subseteq \mathbb{C}$ intersecting $\sqrt[m+1]{1}$ at end points gives $S_\delta \subseteq M_m$ a matching cycle.

ex.:



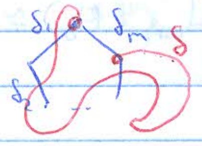
$\rightsquigarrow S_i = S_{\delta_i} = i^{\text{th}}$ 0-section of the plumbing

$S_{k,l} = \tau_{S_{k+1}} \circ \dots \circ \tau_{S_{l-1}} (S_l)$, $k < l$, as



$\to S_{12} = \tau_{S_2}^{-1} S_1$.

Definition: E_δ^{2n+2} = Lefschetz fibration with fiber M_m^{2n} and $m+1$ vanishing cycles $V_1 = S_1, \dots, V_m = S_m$, and $V_{m+1} = S_\delta$, δ some path.

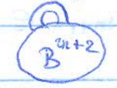


What does this E_δ^{2n+2} look like?

Property: $\forall \delta$, E_δ is diffeomorphic to a $(n+1)$-bundle over S^{n+1}.

Proof: 1) Attaching handles along $V_1, \dots, V_m$ kills the topology of the fiber $\rightsquigarrow B^{2n+2}$.

2) Attach another handle to B^{2n+2} along V_{m+1}:



$\square$

Proposition: (n even) if δ can be isotoped in $\mathbb{C}$ to $\delta_{k,l}$ crossing an even number of $\sqrt[m+1]{1}$, then E_δ is almost symplectomorphic to $(T^*S^{n+1}, \omega_{std})$.

$\hookrightarrow$ pullback of almost - $\mathbb{C}$-str is in the same homotopy class ~~of~~ than the one we had.

(n odd) then the same holds if $[V_{m+1}] = \sum_{i=k}^{l-1} [V_i] \in H_n(M_m)$.

Theorem A: if $\delta = \delta_{k,\ell}$ for some $k<\ell$, then E_δ is sympl. to $(T^*S^{n+1}, \omega_{std})$.

Theorem B: if $\delta \neq \delta_{k,\ell}$ for all k,ℓ, then E_δ has no exact Lagrangian L with $[L] \neq 0$ in $H_{n+1}(E_\delta) \cong \mathbb{Z}$.

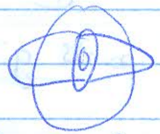
$\rightsquigarrow$ E_δ is not symplectomorphic to $(T^*S^{n+1}, \omega_{std})$.

Rem: • Implies that $\exists$ infinitely many δ that give exotic symplectic structures on $(T^*S^{n+1}, \omega_{std})$

• Unclear from the proof whether E_δ and $E_{\delta'}$ are symplectomorphic for $\delta, \delta' \neq \delta_{k,\ell}$.

Proof of A:

• Consider first $\delta = \delta_m$: . Then S_δ and S_m (corresponding matching cycles) are hamiltonian isotopic. So, we attach 2 handles whose union of cores form a Lagrangian sphere.



The other ones kill the topology. So we get T^*S^{n+1}.

If $\delta \neq \delta_m$, apply some Hurwitz moves to assume $\delta = \delta_i$ for some i, and same as before: all others kill the topology, and we are left with a Lagrangian sphere. □

Proof of B: rest of this talk.

Assume we have an $L \subseteq E_\delta$, $[L] \neq 0 \in H_{n+1}(E_\delta) \cong \mathbb{Z}$. We will prove that $\delta = \delta_{k\ell}$ for some $k<\ell$.

• $E_\delta \to \mathbb{C}$ with $p_i \in \mathbb{C}$ critical values

$\gamma_i \subseteq \mathbb{C}$ vanishing paths for v_i

$\Delta_i \subseteq E_\delta$ thimbles for γ_i.

• $\Delta_{m+1} \in H_{n+1}(E, \partial E) \cong H^{n+1}(E)$ (by Poincaré-Lefschetz duality)

$\cong \mathbb{Z}$ is a generator,

and $[L] \neq 0 \Rightarrow L \cdot \Delta_{m+1} \neq 0$.

$H_{n+1}(E_\delta) \otimes H_{n+1}(E_\delta, \partial E_\delta) \longrightarrow \mathbb{Z}$ is non-degenerate.

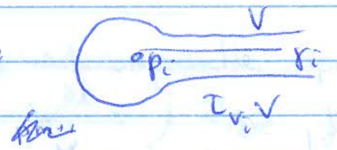
Proposition: L closed exact Lagrangian with $L \cdot \Delta_{m+1} \neq 0$.

$\Rightarrow HW(\Delta_{m+1}, \Delta_{m+1}) \neq 0$.

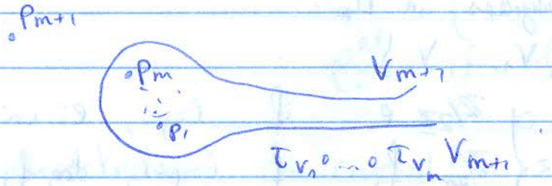
Proof: If $HW(\Delta_{m+1}, \Delta_{m+1}) = 0$, then $HW(\Delta_{m+1}, L) = 0$ since it is a $HW(\Delta, \Delta)$-module.
$\rightsquigarrow L \cdot \Delta = \chi(HW(\Delta, L)) = 0$, as if one Lagrangian is compact, HW is just the ordinary HF. □

2) Spectral sequences:

Monodromy is τ_v :



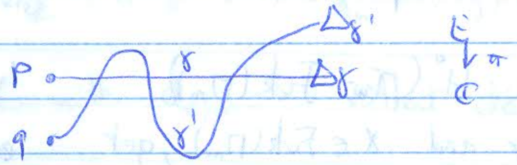
and



Counting J-holomorphic sections with Lagrangian boundary gives $\sigma \in HF(V_{m+1}, \tau_{v_1} \circ \dots \circ \tau_{v_m} V_{m+1})$.

Goal: $HW(\Delta_{m+1}, \Delta_{m+1}) \neq 0 \Rightarrow \sigma = 0$.

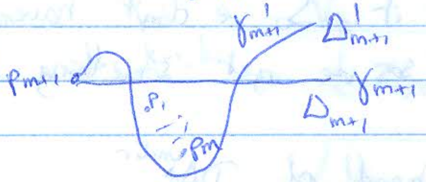
Spectral sequence:



$$E_1 = \bigoplus_{x \in \gamma \cap \gamma'} HF(\Delta \cap E_x, \Delta' \cap E_x) \Rightarrow HF(\Delta_\gamma, \Delta_{\gamma'}).$$

That's because J-hol curves in E_δ can only happen in fibers (project to $\mathbb{C}$, and apply max principle in the base)

Proposition: $\exists$ S.S. $\mathbb{Z}/2\mathbb{Z} \oplus HF(V_{m+1}, \tau_{v_1} \circ \dots \circ \tau_{v_m} V_{m+1}) = E_1^{1,*}$
$\oplus HF(V_{m+1}, V_{m+1}) = E_1^{2,*}$
converging to $HF(\Delta_{m+1}, \Delta'_{m+1})$.



- $\mathbb{Z}/2\mathbb{Z}$ given by p_{m+1} (only $\cap$ pt of thimbles).
- $HF(V_{m+1}, \tau_{v_1} \circ \dots \circ \tau_{v_m} V_{m+1})$ given by the $\cap$ points.
- $HF(V_{m+1}, V_{m+1})$ given by ~~p_{m+1}, which is~~ Δ_{m+1}

$d^1 : E_1^{1,*} \to E_1^{2,*}$ has dual

$$HF(V_{m+1}, \tau_{V_1} \circ \dots \circ \tau_{V_m} V_{m+1}) \xleftarrow{\circ \sigma} HF(V_{m+1}, V_{m+1})$$

- if $\sigma \neq 0$, then $HF(\Delta_{m+1}, \Delta'_{m+1}) \neq 0$ by some delicate rank computations.
- $\underset{\overset{\shortparallel}{0}}{HF(\Delta_{m+1}, \Delta'_{m+1})} \longrightarrow \underset{\overset{\shortparallel}{0}}{HW(\Delta_{m+1}, \Delta_{m+1})}$.

So, we showed that $\sigma = 0$.

3) Fukaya categories:

- $Fuk(M_m)$ = closed Lagrangians in M_m.
- Let A^{dir} with Objects $= \{V_1, \dots, V_m\}$

$$hom(V_i, V_j) = \begin{cases} \mathbb{Z}/2\mathbb{Z}\, e_i & \text{if } i = j, \ e_i \text{ unit} \\ \mathbb{Z}/2\mathbb{Z}\, f_j & \text{if } i = j-1, \ \deg f_j = 1 \\ 0 & \text{otherwise} \end{cases}$$

~~Notation~~ And $\mu^2(e_i, e_i) = e_i$

$\mu^2(e_i, f_i) = f_i = \mu^2(f_i, e_i)$.

For degree reasons, $\mu^k = 0$ if $k \neq 2$.

- $DFuk(M_m) := H^0(Tw\ Fuk(M_m))$
- For S^n Lagr and $X \in Fuk(M_m)$, get

$$\begin{tikzcd} X \arrow[r, "\sigma"] & \tau_{S^n} X \arrow[dl] \\ Hom(S^n, x) \otimes S^n \arrow[u, "ev"] & \end{tikzcd}$$

If $\sigma = 0$, then $X \oplus \tau_{S^n} X \cong Hom(S^n, x) \otimes S^n$.

- In our situation,

$$\begin{tikzcd} V_{m+1} \arrow[r, "\sigma"] & \tau_{V_1} \circ \dots \circ \tau_{V_m} V_{m+1} \arrow[d] \\ & C \arrow[ul] \end{tikzcd}$$

C = twisted complex of $V_1, \dots, V_m \in \mathcal{D}(A^{dir})$.

So, $\sigma = 0 \Rightarrow V_{m+1} \oplus \tau_{V_1} \circ \dots \circ \tau_{V_m} V_{m+1} \cong C \in DA^{dir}$.

Rem: here $\sigma = 0$, not some power of it. So we don't have to iterate the composition of Dehn twists, so we get something directed.

Rem: any $L \in M_m$ is a direct summand of DA^{undir} cf. previous remark.

4) Quiver representations:

Directed A_m-quiver: $\bullet \to \bullet \to \cdots \bullet \to \bullet$ with m vertices.

Definition: a representation of a $\overset{\text{directed } A_m}{\vee}$ quiver is $W_1, \dots, W_m$ $\mathbb{Z}/2\mathbb{Z}$ vector spaces and $\rho_i : W_i \to W_{i+1}$.

Gabriel's theorem: any indecomposable repr. is isomorphic to $W^{k,l}$ for some k, l, where
$$W_i^{k,l} = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{for } k < i < l \\ 0 & \text{otherwise} \end{cases}$$
$$\rho_i^{k,l} = \begin{cases} id & \text{for } k < i < l \\ 0 & \text{otherwise.} \end{cases}$$

Rem: directedness is crucial!

- The objects of $Tw\, A^{dir}$ are $C = \bigoplus W_i \otimes V_i$, $W_i = \mathbb{Z}/2\mathbb{Z}$ vector space and $V_i \in A^{dir}$, and d_C is an endomorphism with
$$d_{C,ji} \in Hom_{\mathbb{Z}/2\mathbb{Z}}(W_i, W_j) \otimes Hom_{A_m}(V_i, V_j)$$
- We can restrict to C with $d_{C,ii} = 0$ without changing the quasi-equivalence type of DA^{dir}.

  $\rightsquigarrow$ only comp. of d_C is $d_{C,i+1,i} = \underset{\hookrightarrow \in Hom_{\mathbb{Z}/2\mathbb{Z}}(W_i, W_{i+1})}{\rho_i} \otimes f_i$

  $\rightsquigarrow$ it is a repr. of a direct A_m-quiver.

Proposition: any indecomposable object of $Tw\, A^{dir}$ is isom. to $W^{k,l}$ (by Gabriel's theorem).

Rem:
- $W^{k-1,k} = V_k$
- $W^{k-1,k+1} = \{ \mathbb{Z}/2\mathbb{Z}^{k} \xrightarrow{id} \mathbb{Z}/2\mathbb{Z}^{k+1} \}$

  $= \{ \underset{\overset{\wr\wr}{\mathbb{Z}/2\mathbb{Z}}}{Hom(V_k, V_{k+1})} \otimes V_k \xrightarrow{ev} V_{k+1} \} = T_{V_k} V_{k+1}$
- Same computation gives in general $W^{k,l} = T_{V_{k+1}} \circ \dots \circ T_{V_{l-1}} (V_l) \cong S_{k,l}$.

End of proof of theorem B:

- E_δ has a Lagrangian L with $[L] \neq 0$ in $H_{n+1}(E_\delta)$
  $\Rightarrow HW(\Delta_{m+1}, \Delta_{m+1}) \neq 0 \quad \Rightarrow \sigma = 0$
  $\Rightarrow V_{m+1}$ is a summand of $C \in DA^{dir}$
- $V_{m+1} \oplus X = C$,
$$\begin{tikzcd} V_{m+1} \arrow[r, hook] \arrow[rr, bend right, "Id"'] & C \arrow[r, "\pi"] & V_{m+1} \end{tikzcd}$$

(*) so $\phi : HF(C, V_{m+1}) \otimes HF(V_{m+1}, C) \longrightarrow \underset{\underset{H(S^n)}{\wr\wr}}{HF(V_{m+1}, V_{m+1})}$ hits $Id \in H^0(S^n)$.

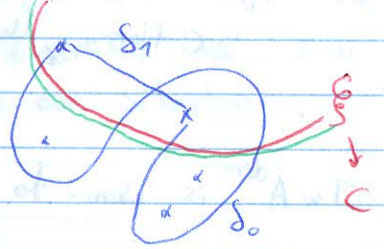
- $C \in DA^{dir} \rightsquigarrow C = \bigoplus S_{k,l}$, as it is a sum of irreducible representations

Proposition: if $\delta_0, \delta_1 \subseteq \mathbb{C}$ are matching paths in
$$\begin{tikzcd} T^*S^{n-1} \arrow[r, hook] & \Pi_m \arrow[d] \\ & \mathbb{C} \end{tikzcd}$$
which are non isotopic, then
$$Im\left(HF(S_{\delta_1}, S_{\delta_0}) \otimes HF(S_{\delta_0}, S_{\delta_1}) \longrightarrow \underset{\underset{H(S^n)}{\wr\wr}}{HF(S_{\delta_0}, S_{\delta_0})}\right) \subseteq H^n(S^n).$$

- So, if $\delta_{m+1} \neq \delta_{k,l}$ for all k,l, then $Im\,\phi \subseteq H^n(S^n)$; contradiction with above (*).

Proof of proposition: if δ_0, δ_1 are non isotopic, then $\exists$ separating path ξ such that $\xi \cap \delta_1 = \phi$, $\xi \cap \delta_0 \neq \phi$

$$\begin{tikzcd} T^*S^{n-1} \arrow[r, hook] & \Pi_m \arrow[d] \\ & \mathbb{C} \end{tikzcd}$$



$\dim HF(L, S_{\delta_1}) = 0$
$\dim HF(L, S_{\delta_0}) = 2\, I(\delta_0, \xi) > 0.$

If $\exists\, a_2 \in HF(S_{\delta_1}, S_{\delta_0})$, $a_1 \in HF(S_{\delta_0}, S_{\delta_1})$ s.t. $a_2 \otimes a_1 \to Id$, then
$$\begin{tikzcd} HF(L, S_{\delta_0}) \arrow[r, "\cdot a_1"] \arrow[rr, bend right, "id"'] & HF(L, S_{\delta_1}) \arrow[r, "\cdot a_2"] & HF(L, S_{\delta_0}) \end{tikzcd}$$

So, $0 < \dim HF(L, S_{\delta_0}) \leq \dim HF(L, S_{\delta_1}) = 0$. □

Ailsa Keating - Homologeous recombinations à la Abouzaid-Seidel:

Theorem: if E is a complex affine variety with $\dim_{\mathbb{R}} \geq 12$, and $q \in \mathbb{N}$. Then, $\exists$ complete finite type Liouville $\tilde{E}$, almost symplectomorphic to E, such that

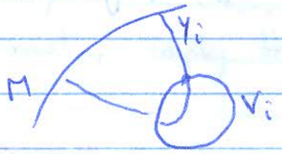
- if $\mathrm{char}(\mathbb{k}) | q$, $SH(\tilde{E}) = 0$.
- if $\mathrm{char}(\mathbb{k}) \nmid q$, $SH(\tilde{E}) = 0 \Leftrightarrow SH(E) = 0$.

Rem: finite type Liouvilles are those that appear as completion of Liouville domains.

1) Dual paths for vanishing cycles.

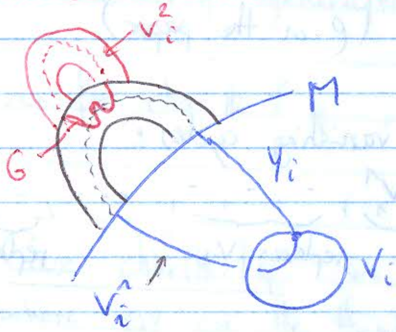
Claim: $\exists$ Lefschetz fibration E with fiber M^{2n}, vanishing cycles $(V_1, \dots, V_m)$ s.t. $\exists$ properly embedded Lagrangian discs $\underbrace{Y_1, \dots, Y_m \subset M}_{\text{with Legendrian boundary}}$ such that $\underbrace{Y_i \cap V_i = \{pt\}}_{\hookrightarrow \text{ hence "dual"}}$

Sketch: Pick $\begin{array}{c} E \\ \downarrow \pi \\ \mathbb{C} \end{array}$ generic, and $\begin{array}{ccc} X & \subset & E \\ \downarrow & & \downarrow \pi \\ pt & \in & \mathbb{C} \end{array}$



vanishing cycles for π $\Leftrightarrow$ matching paths for π.

2) Handle attachment, x 2:



For V_i^2: instead of using the cocore, use some wacky embedded disc G (which we construct on the next page)

$V_i^1 = Y_1 \cup$ core ; it's a sphere
$V_i^2 = G \cup$ core ; it's a sphere too.

Suppose $n \geq 5$, $q \in \mathbb{N}$. Cone $(S^1 \xrightarrow{q} S^1)$ is the mod q ~~is a~~ Moore space.

Fact: this embeds into S^4 (believe it)

$\rightsquigarrow$ get $\mathcal{U}_- \subseteq S^{n-1}$, with $\mathcal{U}_- \underset{\text{homotopy}}{\simeq}$ cone $(S^1 \xrightarrow{q} S^1)$.

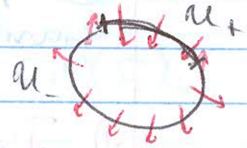
↳ thickening of the cone to make it codim 0.

S^{n-1} decomposes as $\mathcal{U}_+ \cup \mathcal{U}_-$, where

- $\mathcal{U}_+$ and $\mathcal{U}_-$ are both domains
- $\partial \mathcal{U}_+ = \partial \mathcal{U}_-$, smooth
- $\chi(\mathcal{U}_-) = 1$

(*) $\tilde{H}^*(\mathcal{U}_-; \Bbbk) = 0 \iff$ char $\Bbbk \nmid q$.

Now, pick $g : \mathbb{D}^n \to \mathbb{R}$ with no critical points on the boundary, such that ∇g points $\begin{cases} \text{inward along Int } \mathcal{U}_+ \\ \text{outward on Int } \mathcal{U}_- \end{cases}$:



Let $G = \text{graph}(dg) \subseteq \mathbb{D}^{2n} \subseteq T^*\mathbb{D}^n$.

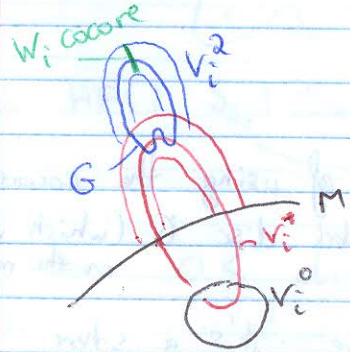
By manipulating g, we can arrange that $\partial G = G \cap S^{2n-1}$ is a Legendrian, and $\partial G \cap (\text{zerosection} \cap S^{n-1}) = \phi$.

Properties 1) $HF^*(G, \text{zero-section}) \cong H\text{Morse}(g)$
$\cong HF^*(\mathbb{D}^n, \mathcal{U}_-) \cong \tilde{H}^*(\mathcal{U}_-)$. Remember (*)

2) $\exists$ smooth isotopy rel ∂ which makes G disjoint from zero-section.

$\deg(dg) = \pm \chi(H^*(\mathbb{D}^n, \mathcal{U}_-)) = 0$

↳ Can construct a pushoff.



E: M, $(V_1, \dots, V_m)$

Pick any $k \in \mathbb{N}$ (ρ in the paper)

$\tilde{E}(k)$: M, $3m$ vanishing cycles:

$(\underbrace{\tau^k_{V_1^2} \tau_{V_1^1} V_1^0, V_1^2, V_1^1}_{\text{replaces } V_1}, \underbrace{-, -, -}_{\text{replaces } V_2}, \dots, \underbrace{-, -, -}_{\text{replaces } V_m})$

- **k=1:** $(\tau_{V^2}^1 \tau_{V^1} V^0, V^2, V^1)$
  $\to (V^2, \tau_{V^1} V^0, V^1)$ by Hurwitz move
  $\to (V^2, V^1, V^0)$ Hurwitz.
  Destabilize: $\tilde{E} = E$.

- **k arbitrary:** recall that we can make G disjoint from the 0-section (part of V_i^1). So, topologically, $V_i^2 \cap V_i^1 = \phi$. That implies that, as almost symplectic manifolds, $\tilde{E}(k)$ is independant of k.
  That's because the Dehn twist along V_i^2 doesn't ~~count~~ change anything then (smoothly), but we can show that the almost symplectic structures coincide too.

Alright. We still have to show that the part of the theorem about SH^* is true.

Case char($\mathbb{k}$) $\nmid q$: $HF^*(V_i^2, V_i^1) = 0$, because it is $\tilde{H}^{*-1}(\mathcal{U}_-, \mathbb{k})$ which is $0 \Leftrightarrow$ char $\mathbb{k} \nmid q$.

Consider the collection of vanishing cycles on $\tilde{E}(k)$, thought of as a full subcategory of $Fuk(\tilde{M}(k))$.
It is independant of k. ↳ fiber of $\tilde{E}(k)$.
Why? $HF^*(V_i^2, V_i^1) = 0$, along with the long exact sequence of a Dehn twist, imply that $\tau_{V^2}^k \tau_{V^1} V^0$ does not see k algebraically.

Proposition (property 4): let E with fiber Π and van. cycles $V_1, \dots, V_m$.
$SH^*(E)$ only depends on the quasi-isomorphism type of the full subcategory generated by vanishing cycles. □ (just algebra)

so for char $\mathbb{k} \nmid q$, $SH^*(\tilde{E}) = 0 \Leftrightarrow SH^*(E) = 0$.

Case char($\mathbb{k}$) $| q$: we will use $SH^*(\tilde{E}(k)) = 0 \Leftrightarrow HW^*(\Delta_i^j, \Delta_i^j) = 0$, where Δ_i^j are the thimbles corresponding to the vanishing cycles of E.
(there is a j because we replaced each of them by 3 things).

Property 2: let F be the total space of a Lefschetz fibration with fibre N, vanishing cycles $(L_1, \dots, L_\ell)$ and thimbles $(\Delta_1, \dots, \Delta_\ell)$.
F': N, $(L_2, \dots, L_\ell)$, thimbles $(\Delta_2', \dots, \Delta_\ell')$.

$$\left. \begin{array}{l} HW_F^*(\Delta_1, \Delta_1) = 0 \\ HW_{F'}^*(\Delta_i', \Delta_i') = 0 \quad \forall i \geqslant 2 \end{array} \right\} \Rightarrow HW_F^*(\Delta_i, \Delta_i) = 0 \quad \forall i.$$

Proof:

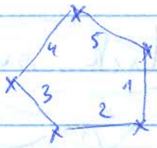
$$(+) \left\{ \begin{array}{l} \text{Recall we have } rk(HF^*(V_i^2, V_i^1)) = rk(\tilde{H}^{*-1}(\mathcal{U}_-; \Bbbk)) =: A. \\ \text{Then } \text{~~Also~~}, \ rk(HF^*(\tau^k_{V_i^2} \tau_{V_i^1} V_i^0, \underset{\uparrow \text{ cocore of } 2^{nd} \text{ handle}}{W_i})) = k.A \qquad (\text{know } HF(V_i^2, W_i)\text{; Dehn twist } k \text{ times}) \\ \text{Also, } rk(HF^*(V_i^2, W_i)) = 1. \end{array} \right.$$

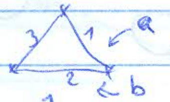
* If $HW^*(\Delta_1^0, \Delta_1^0) \neq 0$, (+) leads to a contradiction. This is because some inequality of HW/HF rank would be satisfied (the same from Maydanskyi/Seidel paper). But it is not true for $k > 1$.

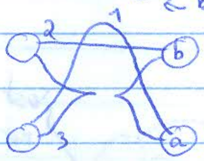
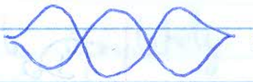
* $HW^*(\Delta_1^1, \Delta_1^1) = HW^*(\Delta_1^2, \Delta_1^2) = 0$ can be argued completely by hand. Then, move to Δ_2^0, etc. This is done by induction. □

Emmy Murphy - A dictionary for Lefschetz bifibrations and Legendrian fronts II

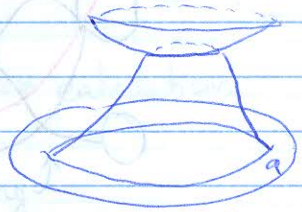
This is a continuation of Roger's talk.

ex 1: $\{xy^k = z^2 + w^2 + 1\}$; it gives  $k+1$-gon as A_n-diagram.

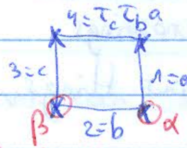
- Take a 3-gon: (k=2)  standard basis: $1 = a$, $2 = b$, $3 = \tau_b a$.

We get (go backwards) . Can cancel 1 with a, and 2 with b: Get  ≅ ,

which is loose.

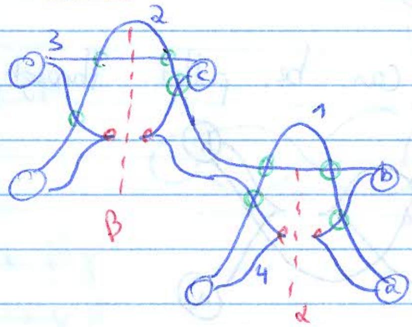
Rotate it: 

Loose means , but it's good enough if we can just find something embedded like that.

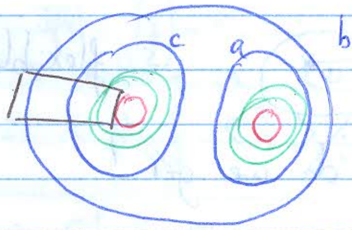
- k = 3: 

α, β Reeb chords

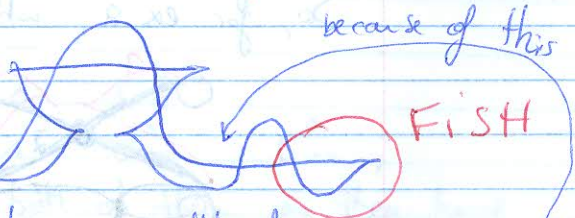
No Reeb chord because a & c do not intersect



(surface in 3D)
(side view)

In more dimensions: 

(top → down view)
red = set of cusps

We can cancel, Reidemeister, etc: 

We can cancel  on the left, but only in a nbhd of the black slice above, because we do not have a rotational symmetry anymore.

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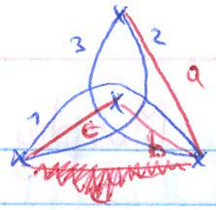
{a,b,c} : linear basis

ex 2: $\{x(xy^2+1)=z^2\}$ $\rightsquigarrow$



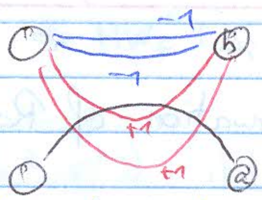
$1 = \tau_c b$

$2 = \tau_c^{-1} \tau_b a$

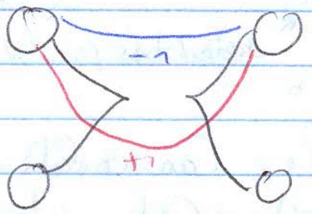
$3 = \tau_b^2 a$

But we don't know how to draw what a square Dehn twist looks like ☹

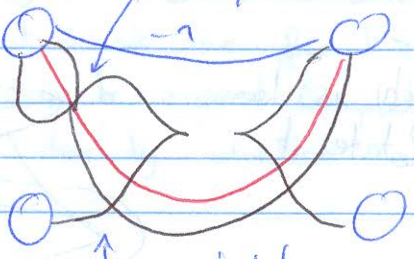
$(id, \tau_b^2 a) \leftarrow$ Leg'n.



Handleslide black over ~~first~~ lower blue; then lower blue and upper red cancel.



Do more handleslides, with black & red one, on the left

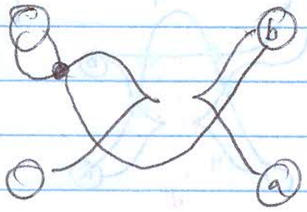


just a cone there; not symmetric

Denote cones by a solid dot.

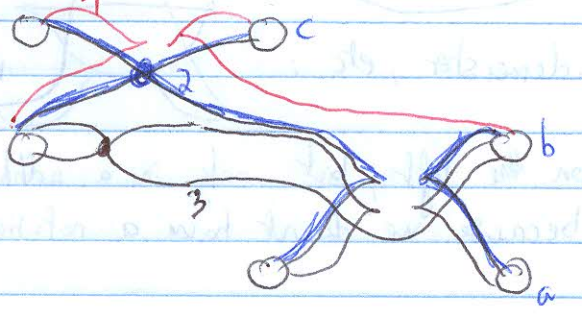
an actual line of crossings

Now the red can be pulled through various things, and we are left with

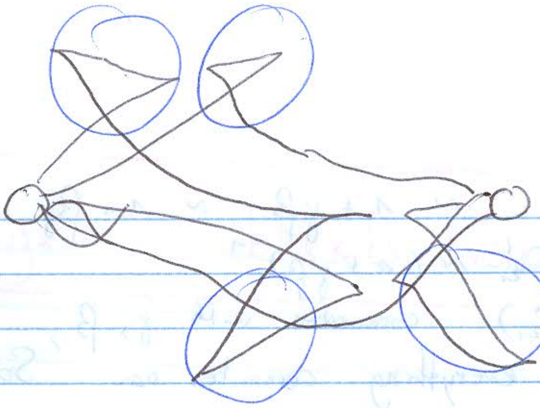


[exercise:  is flexible for any order]

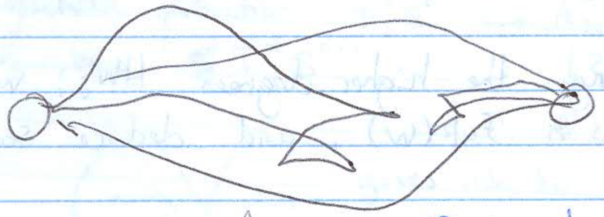
So, for ex 2, what do we get?



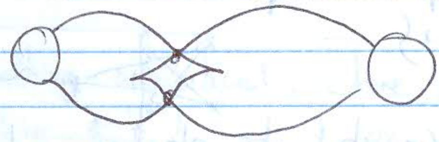
Cancel a and c :



↓ Reidemeister on circled areas

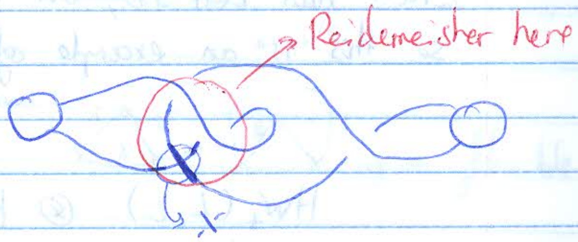


↓ more Reidemeisters

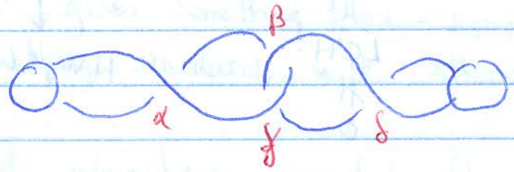


genuinely completely symmetric

Now, dim 4 LCH calalations:



which we can simplify to



$\rightarrow d\alpha = 1 + c^0_{12} + \beta\gamma$

$d\delta = 1 + c^0_{12} + \gamma\beta$

$|\alpha| = |\delta| = 1$; $|\beta| = |\gamma| = 0$, $|c^0_{12}| = 0$.

Notice that there is nothing in <0 degree

So $LCH_0(\Lambda) = \langle \beta, \gamma, c^0_{12} \rangle / im(d)$

What about the c^p_{ij}? In grading 0: c^0_{12}

" " 1: c^3_{22} , c^1_{11} ,

but we dont need to keep track of both of them, as they have the same differential $1 + c^0_{12} c^1_{21}$

So c^0_{12} , c^1_{21} are inverse of each other.

$C^0_{12} \sim 1 + \gamma\beta \sim 1 + \beta\gamma$, $\gamma\beta \sim \beta\gamma$

So $C^1_{21} \sim (1+\gamma\beta)^{-1}$

$(C^1_{21})^{-1}$ commutes with γ, β, so C^1_{21} also.

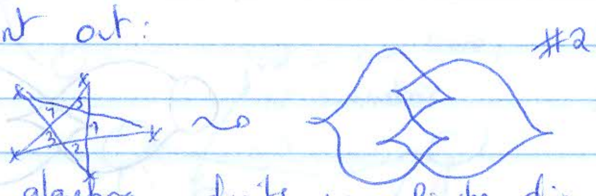
So, everything commutes now. So we get

$$\begin{tikzcd} LCH_0 \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong \text{ BEE}"] & \mathbb{F}[\beta, \gamma, (1+\beta\gamma)^{-1}] \\ HW_0(\text{Cocore}) & \end{tikzcd}$$

Computing the higher degrees HW_i, we could maybe conclude that this is Fuk(W), and deduce something about the mirror.

Another example to point out:

$\{x^2y^3 = z^2+1\}$:



,

which has LCH $\neq 0$, but its algebra admits no finite dim representation, so this is an example of wrapped Fukaya category $\neq 0$, but no compact Lagrangian.

$$\begin{tikzcd} HW_*(L,L) \arrow[d, phantom, "\cong"] & \otimes & HW_*(L,C) \arrow[d] \\ LCH \arrow[d, phantom, "\neq"] & & HW_*(L,C) \\ 0 & & \end{tikzcd}$$

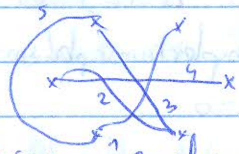
gives a representation.

Kyler Siegel - Subflexible Weinstein manifolds

Recall about Milnor fibers: $A^{2n}_m = \{ z_1^{m+1} + z_2^2 + \dots + z_{n+1}^2 = 1 \} \subseteq \mathbb{C}^{n+1}$.
We have a lefschetz fibration

$$\begin{tikzcd} T^*S^{n-1} \arrow[r, hook] & A^{2n}_m \arrow[d, "z_1"] \\ & \mathbb{C} \end{tikzcd}$$

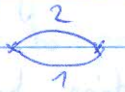
, giving m+1 vanishing cycles $W_1, \dots, W_{m+1} \subseteq T^*S^{n-1}$.

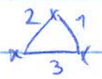
Any A_m-diagram  gives rise to

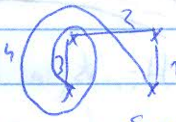
$$\begin{tikzcd} A^{2n}_5 \arrow[r, hook] & X^{2n+2} \arrow[d, "\text{Lefs. fibr.}"] \\ & \mathbb{C} \end{tikzcd}$$

, which has vanishing cycles $V_1, \dots, V_5 \subseteq A^{2n}_5$.

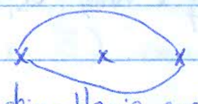
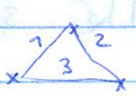
ex: if we draw something subcritical, where every handle can be cancelled, we get nothing (ie $\mathbb{C}^{n+1}$). ex: 

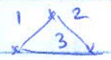
ex:  gives T^*S^{n+1}.

ex:  gives T^*S^{n+1}, but  ~  gives flexible T^*S^{n+1}

ex: (cf Oleg's talk):  gives something quasi-symplecto to something standard, with $SH = 0$

Question: are these kinds of Weinstein manifold flexible?

ex:  $\xrightarrow{\text{attach a critical handle}}$  $\xrightarrow{\text{Hurwitz move}}$  flexible!
(Maydanskiy-Harris eye)

Viterbo: SH () $\longrightarrow$ SH (),
$\|$
0

So by the $1=0$ argument, SH() $= 0$

Question: is it flexible?

Twisted symplectic cohomology: "B-field"

(W, λ) Liouville domain, Ω closed 2-form on W.

Usual SH differential: $d(\gamma_+) = \sum_{u \in \mathcal{M}_0(\gamma_-, \gamma_+)} \gamma_-$, counting 

Twisted differential: $d(\gamma_+) = \sum_{u \in \mathcal{M}_0(\gamma_-, \gamma_+)} t^{\int u^* \Omega} \gamma_-$. Call it $\widetilde{SH}(W, \lambda; \Omega)$

Facts:
- only depends on $[\Omega] \in H^2(W; \mathbb{R})$
- invariant under symplectomorphisms of $(\hat{W}, \hat{\lambda})$ preserving $[\Omega]$.
- still $\widetilde{SH}$(flexible) $= 0$.

Subflexibilization:

Let $X = \mathrm{Lef}(W, V_1, \dots, V_k)$ be a Weinstein Lefschetz fibration with fiber W and vanishing cycles $V_1, \dots, V_k \subseteq W$.

Assume: $T_1, \dots, T_k$ Lagrangian discs with Legendrian boundary such that $|T_i \pitchfork V_i| = 1$
(T_i can touch the other ones)

$W' = W \cup H_1 \cup \dots \cup H_k$, where H_i is a critical Weinstein handle attached along ∂T_i.

$S_i = T_i \cup (\overset{\text{core}}{\text{of}}\ H_i)$ is a Lagrangian sphere in W

The subflexibilization of X is $X' := \mathrm{Lef}(W', \tau_{S_1}^2 V_1, \dots, \tau_{S_k}^2 V_k)$.

Proposition: X' is subflexible.

Proof: attach k critical handles $\rightsquigarrow \mathrm{Lef}(W', \tau_{S_1}^2 V_1, S_1, \dots, \tau_{S_k}^2 V_k, S_k)$.
apply k Hurwitz moves $\rightsquigarrow \mathrm{Lef}(W', S_1, \tau_{S_1} V_1, \dots, S_k, \tau_{S_k} V_k)$.
(so we get something Weinstein homotopic)
exercise: this is flexible.

Corollary: $SH(X') = 0$, by the Viterbo argument again.

Theorem: for $\dim X = 6$, $\widetilde{SH}(X', \Omega) \cong SH(X)$ for some 2-form Ω.

ex: subflex() $\rightsquigarrow$ , THE EYE.

So $\widetilde{SH}($ , $\Omega) \cong SH(T^*S^3) \cong H(\mathcal{L}S^3) \neq 0$,
so  is not flexible.

ex: subflex (Mclean's exotic $\mathbb{C}^3$) gives a non flexible Weinstein sublevel set of $(\mathbb{C}^3, \lambda_{std})$, which is flexible.

Corollary: every flexible Weinstein domain has, after a Weinstein homotopy, a non flexible sublevel set.
(since we can do it for $\mathbb{C}^3$ as above, + more reflexion on higher dimensions).

How to prove the theorem?

Main computational tool:

Recall: $X = Lef(W; V_1, \dots, V_k)$
- Slogan: "$SH(X)$ can be computed from $Fuk(V_1, \dots, V_k)$"; there is some formula.
- Twisted version: let Ω be a closed 2-form on W with support disjoint from $V_1, \dots, V_k$. Let $\tilde{\Omega}$ be the pullback of Ω to X (ie extend to handles by 0; that's why we need disj. from V_i)
"$\widetilde{SH}(X, \tilde{\Omega})$ can be computed from $\widetilde{Fuk}(V_1, \dots, V_k; \Omega)$.

(standard Fuk  $\rightsquigarrow$ twisted Fuk $t^{\int_u \omega}$ )

As before, $W' = W \cup H_1 \cup \dots \cup H_k$. With W 4-dimensional.

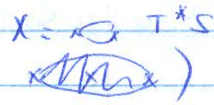
Main proposition: (1) $V_i \not\simeq \tau_{S_i}^{\text{mod } 2} V_i$ in $Fuk(V_1, \dots, V_k)$ (not quasi-isom)
(2) $V_i \simeq \tau_{S_i}^2 V_i$ in $\widetilde{Fuk}(V_1, \dots, V_k; \Omega)$, for Ω to be defined shortly

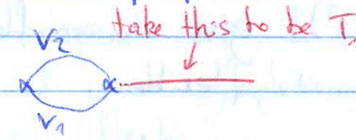
Proof of the theorem:

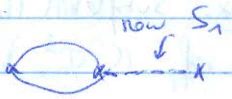
$\widetilde{Fuk}(\tau_{S_1}^2 V_1, \dots, \tau_{S_k}^2 V_k) \simeq \widetilde{Fuk}(V_1, \dots, \overset{\Omega}{\overset{\vee}{V_k}}; \Omega)$ by previous prop

$\simeq Fuk(V_1, \dots, V_k)$.

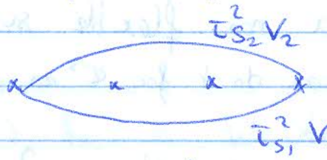
Main tool $\Rightarrow \widetilde{SH}(X'; \widetilde{\Omega}) \cong SH(X)$ $\square$

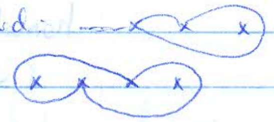
Proof of the main proposition: (in the case of )

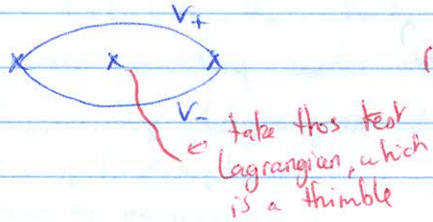
Start with 

$\xrightarrow{\text{attach } H_1}$ 

$\rightsquigarrow$  $\subseteq \tau_{S_1}^2 V_1$, homotopic to the eye

Let's keep going: $\overset{\text{attach } H_2}{\rightsquigarrow}$ 

we did: 

Take 

(1) we have $\begin{cases} HF(V_-, T) \neq 0 \\ HF(V_+, T) = 0 \end{cases}$

$\Rightarrow V_- \not\simeq V_+$.

(2) Consider the product map $HF(V_+, V_-) \otimes HF(V_-, V_+) \xrightarrow{\pi} HF(V_-, V_-)$

$\cong \Bbbk\langle e, f\rangle$

identity "fundamental class"

- f is always in the image of π (PD for HF).
- e is in the image of $\pi \Leftrightarrow V_- \simeq V_+$ (Keating)

π counts rigid pseudo-holomorphic triangles .

MB techniques $\rightsquigarrow$ count 

passing through a fixed point of V_-

Upshot: the coefficient of e in the image of π is #{sections }

$V_- \simeq V_+ \Leftrightarrow$ # {  } $\neq 0$

$\Leftrightarrow$ # {  } $\neq 0$.

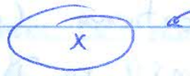
(⚠ we should keep track of where we glue, but by counting degrees of evaluation map, we see that we can forget it)

(Seidel's gluing thm)

$\Longleftrightarrow$ $\#\left\{\text{sections}\right.$  $\left.\right\} \cdot \#\Big\{$  $\Big\} \neq 0$

- Basic fact: $\#\left\{\text{sections}\right.$  $\left.\right\} \neq 0$.
- For the other one $\mathbb{C}^2$ (cf Nati's talk)

$$\begin{tikzcd} \mathbb{C}^2 \arrow[d, "\pi_{std}(z_1,z_2)=z_1^2+z_2^2"] \\ \mathbb{C} \end{tikzcd}$$

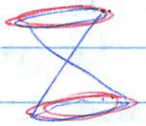
 standard Lagrangian boundary condition

$\bigcup_{z\in S^1} \Sigma_z$, where $\Sigma_z := \{\pm\sqrt{z}\,x \mid x \in S^1 \subset \mathbb{R}^2 \subset \mathbb{C}^2$

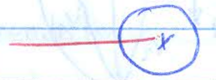
The sections of π_{std} with this boundary condition:

$u_a : \mathbb{D}^2 \to \mathbb{C}^2$, $u_a(z) = az + \bar{a}$

$\|a\|^2 = 1/2$, $\pi_{std}(a) = 0$

$\{ a \in \mathbb{C}^2 \mid \|a\|^2 = 1/2,\ \pi_{std}(a) = 0\} \longrightarrow$  This is a circles C_1, C_2.

$\#\left\{\text{sections}\right.$  $\left.\right\} = 1 - 1 = 0$.

Let Ω be Poincaré dual to a small perturbation of this thimble: 

Fact: $\int u_a^* \Omega = \begin{cases} 0 & \text{if } a \in C_1 \\ 1 & \text{if } a \in C_2 \end{cases}$

With B-field: $\#\left\{\text{sections}\right.$  $\left.\right\} = t^0 \cdot 1 - t^1 \cdot 1 = 1 - t \neq 0$.

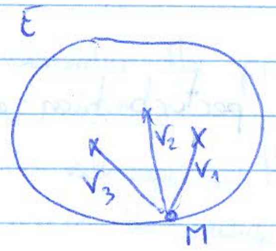
$\square$

Questions and answers session

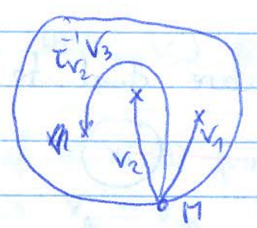
a) Dream of full algebraic structures + relations ... ?
b) LCH computations in higher dimensions ?
c) Are all maps and relations "geometric" ?
d) Degenerations and gluing ... (sections? other operations?) ?
e) Hurwitz moves + stabilizations (define them) ?
f) Generality of A_n-diagrams, eg D_n?
g) SH_{SFT} vs SH_{Ham}?
h) Leg. Kirby calculus?
i) Analytic issues?
j) Why does SFT in char p not necessarily work?

h), e)
(Emmy)

Suppose we start with $Lef(M;(V_1,\dots,V_m))$.



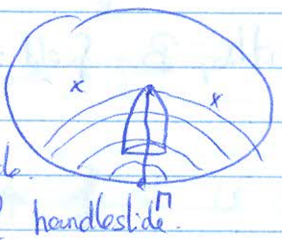
vs



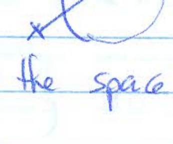
$: Lef(M;(V_1,V_2,V_3)) = Lef(M;(V_1,\tau_{V_2}^{-1}V_3,V_2))$

In general, $\tau_{f^{-1}(\gamma)} = f\,\tau_\gamma\, f^{-1}$, so $\tau_1\tau_{\tau_2^{-1}(3)}\tau_2 = \tau_1\tau_2\tau_3\tau_2^{-1}\tau_2 = \tau_1\tau_2\tau_3$.

Look at the Morse fct $\varphi = dist(\cdot, \Pi)$, defining the Weinstein structure:



As we pass a crit pt, we get a handleslide.
A Hurwitz move is a special case of handleslide$^\Pi$.

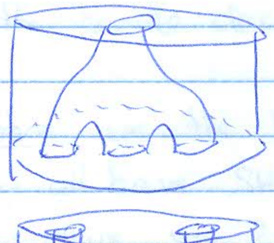
Stabilization: add a  outside. We can see that this does not change the type of the space.

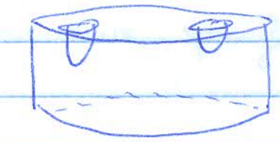
g) (Zack, Sheel) SH_{Ham}: involves curves of Hamiltonian. Let $Y = \partial M$.

$SH_{SFT} \overset{v.s.}{\cong} \left(\mathrm{Morse}(M) \oplus \Bbbk\mathcal{X}(Y) + \Bbbk\mathcal{X}(Y)[1] , \begin{pmatrix} d_{Morse} & c & 0 \\ 0 & \delta_c & \delta_{MB} \\ 0 & D & \delta_c' \end{pmatrix} \right).$

- $\langle \delta_c(x'), y \rangle$ involves counts ~~the~~ of holomorphic cylinders, with some capping off.
- δ_{MB} is usually 0 (Morse-Bott)
- D: to go from min to max  : we can't gradient flow; we have to hit it on the nose.

Rem: curves can escape to $-\infty$ (concave end), but not to $+\infty$ (convex end) because we have a maximum principle. That's why we have to cap.

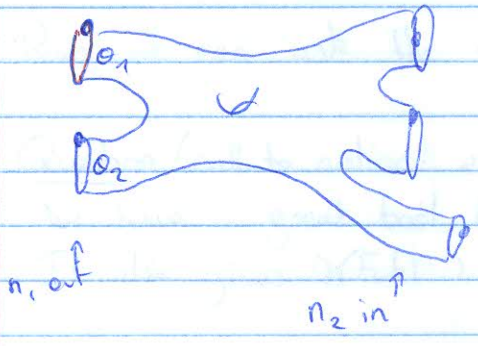




Rem: Ham counts trajectories, but SFT counts actual holomorphic curves, so there is a bigger symmetry group.

Bourgeois - Oancea: $$\begin{tikzcd} SH_{SFT} \arrow[r, "\simeq"] & SH_{Ham, SFT} \arrow[d, phantom, "\wr\|"] \\ & SH_{Ham} \end{tikzcd}$$, assuming that the analytic foundations have been worked out. (chain level map)

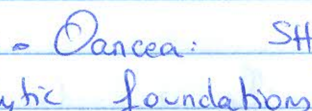
a) (Sheel)



⚠ Choices of asymptotic markers.
We need $n_2 > 0$ (max principle)

We also have an input length $(\ell_1, \dots, \ell_{n_2})$, coming from the length of the Reeb orbit. $\nearrow L$
Restriction: Σ out length $\leq L$.

$\theta_1 = 0 \Rightarrow$ need Morse output.
As output length $\to 0$, we might have some Deligne-Mumford compactification,



circles can collapse.
We should account for surfaces with length markers.

But we don't know how to encode that in Fuk.

i) Which analytic things can we use and not use?

SH_{SFT} has problems, and ham. Floer things too.

Generally, if there is no choice of multiple covers, then we are safe.

⚠ (hol) cylinders, (hol) discs, bubbles, and higher genus things.

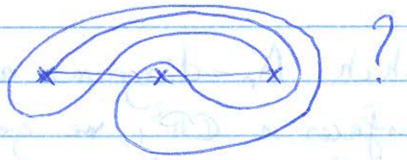
△ unstable phenomena.

Open questions

Question: how to detect flexibility?
Under flexibility hypothesis, everything is understood. But everything has a price, so determining if something is flexible is extremely difficult.

Question: when is a Lefschetz fibration flexible?
It is difficult, even for A_n-type things. A good think to look at is the handle decomposition.

Question: are the Maydanskiy-Seidel examples flexible?
They all have SH=0 (necessary but not sufficient condition.
Even more specific:



Question: when is affine algebraic flexible? Is there an algebraic definition of flexibility?

Rem: we can ask the same questions with "subflexible".

Question/call to action: examples!
We have a great tool to calculate WFuk(A_n-LF): the dictionnary.
It also gives WFuk(..) with homology coefficients. LCH does this well.
↳ Novikov coefficients, ie $H_2(W)$

Question: criteria for existence/non-existence of compact Lagrangians?
Ie, what does "no finite-dim representation" mean geometrically?
We know what's up for flexible or subflexible.

Question: higher dim LCH computations? Mostly done by Ekholm, etc and M. Sullivan - Rutherford. for dim = 5, and a bit more if the set of singularities does not look too bad.

Igor Uljarevic, 1404.2128

Question: LCH detection, relations with fixed point HF.
We have formulae involving WFuk, we we'd like something easier.

$$\begin{tikzcd} (W,\lambda,\varphi) \arrow[r, squiggly] \arrow[loop below, "\mathrm{Symp}"] & \{\text{Legendrian spheres}\} & \pi_0\,\mathrm{Symp} \\ (W,\lambda,\varphi) \arrow[r] & \{\text{different Legendrian spheres}\} \arrow[u, "{\text{handleslides, cancellations}}"'] \end{tikzcd}$$

If W flexible, then $\pi_0\,\mathrm{Symp}(W) \cong \pi_0\,\mathrm{Diff}^J(\cancel{M}W)$ (theorem).
But even under flexibility hypotheses, we do not know anything about $\pi_0\,\mathrm{Symp}_{\substack{\text{compactly}\\ \text{supported}}}(W)$. The isomorphism above doesn't preserve the "compactly supported" hypothesis. Is there anything to say about $\pi_0\,\mathrm{Sym}_{\mathrm{cpct}}(W) \longleftrightarrow \pi_0\,\mathrm{Diff}^J_{\mathrm{cmpt}}(W)$

Question: which A_n-diagrams arise as algebraic?
For hypersurfaces in $\mathbb{CP}^2$, we get a knot at ∞, and the knots that can appear this way have a characterization. What about A_n-diagrams?
Beginning of answer: growth rate of SH ℓ. Mark McLean and Seidel: $\liminf \frac{\log \mathrm{rk}\, SH_*^{[0,L]}}{\log L} \leqslant \dim_{\mathbb{C}}$. What about A_n-diagrams, specifically.

$(\kappa_{\mathrm{Kodaira}} > 0 \Rightarrow SH_{50} \neq 0.)$

Question: is every simply connected flexible W^6 algebraic?
Might answer relatively quickly, as simply connected 6-manifolds are kind of known. They are presumably of A_n-type. If it's true, we can probably construct them by hand.

Question: is every flexible algebraic?

Question: how much is a random (with $\kappa = -\infty$) variety flexible?

Question: an isolated singularity is not flexible because it has vanishing cycle. What about the non-isolated singularities? ie $xy^k - z^2 - w^2$.

Question: dictionnary for more general Lefschetz fibrations (non A_n)? (i.e. $\underset{\text{diagrams}}{D/E}$, tri-fibrations, etc).

Question: $\{x(xy-1) = z^2 + t^3\} = E$. We have

$$\begin{tikzcd} D^4 \arrow[r, hook] & E \arrow[d] \\ & \mathbb{C} \end{tikzcd}$$

,

$E \overset{C^\infty}{\cong} \mathbb{C}^3$, but $E \overset{alg}{\not\cong} \mathbb{C}^3$. (Koras-Russell).

$E \underset{\text{bihol}}{\cong} \mathbb{C}^3$? $E \underset{\text{Symp}}{\cong} \mathbb{C}^3$?

Question: are there algebraic varieties that are deformation equivalent as symplectic, but not through algebraic varieties?

ex: $\{xy^k = z^2 + w^2 + 1\}$ are equivalent through analytic 0-sets (Stein deformation equivalent), but we do not know through polynomial 0-sets.

Or: $\mathbb{C}^4$ vs $\text{contractible}^3 \times \mathbb{C}$.