# MIDTERM 2
# MATH 2030 ODE, FALL 2018

*Instructions:* Solve as many problems or subproblems as you can in the given time. You should show all your work in order to get full or partial credit. You may write on the back of pages and staple additional pages if you run out of space. I recommend first working on whichever problems you know how to solve before spending time on the trickier ones. The raw scores will be curved and you do not necessarily need to solve all the problems to get a good grade. Good luck!!

**Problem 1.** (6 pts / 100) Consider the ODE

$$y'''(t) + sin(t)y'(t) + 17y(t) = 0.$$

Suppose $f(t), g(t), h(t)$ are solutions, defined for all $t \in \mathbb{R}$. Suppose that we have

$$f(\pi) = 1, \quad f'(\pi) = 2, \quad f''(\pi) = 3,$$
$$g(\pi) = 4, \quad g'(\pi) = 5, \quad g''(\pi) = 6,$$
$$h(\pi) = C, \quad h'(\pi) = 8, \quad h''(\pi) = 9,$$

where $C$ is a real constant. Do $f(t), g(t), h(t)$ together form a fundamental set of solutions? *Note: your answer may depend on $C$.*

**Solution 1.** Notice that the ODE is linear and the function $\sin(t)$ is continuous for all $t$. Therefore $f, g, h$ form a fundamental set of solutions if and only if their Wronskian is nonzero for all $t$. The Wronskian at $\pi$ is given by

$$\begin{aligned} W(f,g,h)(\pi) &= 1(5*9 - 6*8) - 2(4*9 - 6*C) + 3(4*8 - 5*C) \\ &= -3 - 72 + 12C + 96 - 15C \\ &= 21 - 3C. \end{aligned}$$

So they form a fundamental set if $C \neq 7$, while if $C = 7$ they do not.

**Problem 2.** (15 pts / 100) Find the general solution to the ODE

$$t^2y''(t) + 2ty'(t) + y(t) = 0, \qquad t > 0.$$

*Note: your solution should be valid for all $t > 0$.*

**Solution 2.** Note that this is an Euler equation, i.e. of the form $t^2y'' + \alpha ty' + \beta y = 0$ with $\alpha = 2$ and $\beta = 1$. Making the ansatz $y(t) = t^r$, we get the indicial equation

$$r(r-1) + 2r + 1 = 0,$$

i.e.

$$r^2 + r + 1 = 0.$$

The roots are therefore $r = (-1 \pm \sqrt{1-4})/2 = -1/2 \pm i\sqrt{3}/2$, which are complex. They correspond to complex-valued solutions

$$\begin{aligned} t^{-1/2 \pm i\sqrt{3}/2} &= e^{\ln(t)(-1/2 \pm i\sqrt{3}/2)} \\ &= t^{-1/2}\left(\cos(\ln(t)\sqrt{3}/2) \pm i \sin(\ln(t)\sqrt{3}/2)\right). \end{aligned}$$

Taking the real and imaginary parts, we get the two linearly independent real-valued solutions $t^{-1/2}\cos(\ln(t)\sqrt{3}/2)$ and $t^{-1/2}\sin(\ln(t)\sqrt{3}/2)$, and therefore the general solution for $t > 0$ is

$$y(t) = C_1 t^{-1/2}\cos(\ln(t)\sqrt{3}/2) + C_2 t^{-1/2}\sin(\ln(t)\sqrt{3}/2).$$

Notice that we're assuming $t > 0$ and otherwise we wouldn't know how to make sense of $\ln(t)$ for $t < 0$. However, we've seen that this solution as actually valid for all $t \neq 0$ provided we replace $t$ with $|t|$.

**Problem 3.** (20 pts / 100) Find the general solution to the ODE

$$y^{(6)}(t) - 6y^{(5)}(t) + 11y^{(4)}(t) - 6y'''(t) = 0.$$

**Solution 3.** This is a sixth order homogeneous ODE with constant coefficients. To solve it, we make the ansatz $y(t) = e^{rt}$, which gives rise to the characteristic equation

$$r^6 - 6r^5 + 11r^4 - 6r^3 = 0.$$

This factors as

$$r^3(r^3 - 6r^2 + 11r - 6 = 0.$$

We need to factor the cubic part $r^3 - 6r^2 + 11r - 6$. The only tool we've really seen is the rational roots theorem, which says that the possible rational roots (if there are any) are $\pm1, \pm2, \pm3, \pm6$. Plugging in 1, we find $(1)^3 - 6(1)^2 + 11(1) - 6 = 0$, so in fact $r = 1$ is a root. Therefore we can write

$$r^3 - 6r^2 + 11r - 6 = (r-1)(r^2 + Ar + B)$$

for some $A, B$. We have

$$(r-1)(r^2 + Ar + B) = r^3 + r^2(A-1) + r(B-A) - B,$$

and therefore we must have $A - 1 = -6$, $B - A = 11$, and $-B = -6$. This means $B = 6$ and $A = -5$. Then the remaining part $r^2 - 5r + 6$ also factors as $(r-2)(r-3)$, and so finally we have

$$r^6 - 6r^5 + 11r^4 - 6r^3 = r^3(r-1)(r-2)(r-3),$$

meaning the roots are $r_1 = r_2 = r_3 = 0$, $r_4 = 1$, $r_5 = 2$, $r_6 = 3$. Recalling that repeated roots give rise to additional solutions by multiplying by $t$ an appropriate number of times, this leads to the general solution

$$y(t) = C_1 + C_2 t + C_3 t^2 + C_4 e^t + C_5 e^{2t} + C_6 e^{3t}.$$

**Problem 4.** (20 pts / 100) Find a particular solution to the ODE

$$y^{(5)}(t) - y'(t) = e^t \sin(t) + 3\cos(t) + t^2 - 1 + e^{-t} + e^{7t}.$$

You may leave your answer in terms of a finite number of undetermined constants, for example $y(t) = A\sin(t) + Be^t$.

**Solution 4.** We will use the method of undetermined coefficients, and the problem states that we don't actually need to find the coefficients. Recall that we make an ansatz that a particular solution has roughly the same shape as the inhomogeneity, in this case $g(t) = e^t \sin(t) + 3\cos(t) + t^2 - 1 + e^{-t} + e^{7t}$, but we need to make sure to include both cosines and sines whenever necessary, and we need to add extra factors of $t$ if happen to encounter a root of the characteristic polynomial. By default we make the ansatz

$$y(t) = Ae^t \sin(t) + Be^t \cos(t) + C\sin(t) + D\cos(t) + E + Ft + Gt^2 + He^{-t} + Ie^{7t}.$$

Now let's check if it needs to be modified. The characteristic polynomial is $r^5 - r = r(r^4 - 1)$, which has roots $0, 1, -1, i, -i$. Note that we would only have to worry about $e^t \sin(t)$ if $1 + i$ were a root, which it isn't. However, $0, i$, and $-1$ correspond to terms in the homogeneity, and they are all single roots. Therefore the modified ansatz should be

$$y(t) = Ae^t \sin(t) + Be^t \cos(t) + Ct\sin(t) + Dt\cos(t) + t(E + Ft + Gt^2) + Hte^{-t} + Ie^{7t}.$$

Note that technically you could add as many extra terms as you want and your answer would still be right, since if you added say $Je^{5t}\cos(17t)$ it would just turn out that $J$ should be zero.

**Problem 5.** (20 pts / 100) Consider the ODE

$$y''(t) + ty'(t) + t^2 y(t) = 0.$$

Given that there is a power series solution of the form $y(t) = \sum_{k=0}^{\infty} a_k t^k$ with $a_0 = a_1 = 3$, find $a_2, a_3, a_4, a_5$.

**Solution 5.** Let's plug in the ansatz $y(t) = \sum_{k=0}^{\infty} a_k t^k$. We get

$$\sum_{k=2}^{\infty} a_k k(k-1)t^{k-2} + \sum_{k=1}^{\infty} a_k k t^k + \sum_{k=0}^{\infty} a_k t^{k+2} = 0.$$

Grouping together all the constant terms, we get $2a_2 = 0$, so $a_2 = 0$. Grouping all the $t^1$ terms, we get $6a_3 + a_1 = 0$, so $a_3 = -1/2$. Grouping all the $t^2$ terms, we get $12a_4 + 2a_2 + a_0 = 0$, so $12a_4 = -3$ and hence $a_4 = -1/4$. Finally, grouping all the $t^3$ terms, we get $20a_5 + 3a_3 + a_1 = 0$, so $20a_5 = -3 + 3/2 = -3/2$ and hence $a_5 = -3/40$.

**Problem 6.** (7 pts / 100) Consider the ODE

$$t(t+1)^3(t+2)^2 y''(t) + t(t+1)(t+2)y'(t) + ty(t) = 0.$$

Find the singular points, and classify them as regular or irregular. *Note: you do not need to justify your answer, although it could help you get partial credit.*

**Solution 6.** If we write the ODE in "stanford form", it looks like

$$y''(t) + \frac{t(t+1)(t+2)}{t(t+1)^3(t+2)^2} y'(t) + \frac{t}{t(t+1)^3(t+2)^2} y(t) = 0.$$

After cancelling common factors, this becomes

$$y''(t) + \frac{1}{(t+1)^2(t+2)} y'(t) + \frac{1}{(t+1)^3(t+2)^2} y(t) = 0.$$

From this we see that $t = -1$ and $t = -2$ are singular points. The point $t = -2$ is regular, but the point $t = -1$ is irregular, since even after multiplying the last term by $(t+1)^2$, we get

$$(t+1)^2 \frac{1}{(t+1)^3(t+2)^2} = \frac{1}{(t+1)(t+2)^2},$$

which is still not analytic at $t = -1$.

**Problem 7.** (12 pts / 100) Consider the ODE

$$2t^2 y''(t) - 3ty'(t) + 2y(t) + 2ty(t) = 0, \qquad t > 0.$$

Given that there are two linearly independent solutions $y_1(t), y_2(t)$ of the form $y_1(t) = t^{r_1}(1 + \sum_{k=1}^{\infty} a_k t^k)$ and $y_2(t) = t^{r_2}(1 + \sum_{k=1}^{\infty} b_k t^k)$, find $r_1$ and $r_2$. *Note: this question is asking you to find the exponents of the singularity at* $t = 0$. *You do not need to find the coefficients* $a_k$, $b_k$.

**Solution 7.** Let's plug in the ansatz $y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$ and try to find the indicial equation. We get

$$\sum_{k=0}^{\infty} 2a_k(k+r)(k+r-1)t^{r+k} - \sum_{k=0}^{\infty} 3a_k(k+r)t^{r+k} + \sum_{k=0}^{\infty} 2a_k t^{k+r} + \sum_{k=0}^{\infty} 2a_k t^{r+k+1} = 0.$$

The lowest power of $t$ appearing is $t^r$, and collecting all these terms give

$$2a_0 r(r-1) - 3a_0 r + 2a_0 = 0.$$

We're assuming $a_0 \neq 0$, so this gives

$$2r^2 - 5r + 2 = 0,$$

and hence roots $r_1 = 2$ and $r_2 = 1/2$.