# MIDTERM 1
# MATH 2030 ODE, SPRING 2018

*Instructions:* Solve as many problems or subproblems as you can in the given time. You should show all your work in order to get full or partial credit. You may write on the back of pages and staple additional pages if you run out of space. I recommend first working on whichever problems you know how to solve before spending time on the trickier ones. The raw scores will be curved and you do not necessarily need to solve all the problems to get a good grade. Good luck!!

**Problem 1.** (20 pts / 100) Consider the following ODE for a function $y(t)$:

$$2t - y(t) + 2y(t)y'(t) - ty'(t) = 0.$$

Find the general solution for $y(t)$. Your answer should involve one arbitrary constant and may be left in implicit form.

**Solution 1.** Notice that this ODE is neither linear nor separable. Let's check if it's exact. We can write this in the form

$$M(t, y) + N(t, y)y' = 0,$$

where $M(t, y) = 2t - y$ and $N(t, y) = 2y - t$. Then we have

$$\partial_y M(t, y) = -1$$
$$\partial_t N(t, y) = -1.$$

Since these agree, the ODE satisfies the condition to be closed, and hence it is exact (note that $M$ and $N$ are continuous for all $t, y$, so no pathologies about "holes" can arise). This means there exists some function $\psi(t, y)$ such that $\partial_t \psi = M$ and $\partial_y \psi = N$. Therefore, by the chain rule the ODE can be rewritten as

$$\frac{d}{dt}\psi(t, y) = 0,$$

and hence the solution is simply $\psi(t, y) = C$ for $C$ an arbitrary constant. In order to find $\psi(t, y)$, we can integrate both sides of the equation $\partial_t \psi = M = 2t - y$ with respect to $t$, holding $y$ fixed, to get

$$\psi(t, y) = t^2 - ty + h(y).$$

Note that the usual constant of integration is instead a function $h(t)$ of $y$, since $\psi$ can be any function whose partial derivative with respect to $t$ is $M$. Taking the partial derivative with respect to $y$, the equation $\partial_y \psi = N$ becomes

$$-t + h'(y) = 2y - t.$$

Luckily, $-t$ on both sides cancels, leaving $h'(y) = 2y$ (of course this is not really luck; exactness precisely guarantees that this will happen). We can therefore take $h(y) = y^2$ (we could also add a constant, but there's no need since we could just absorb it into the constant $C$). In conclusion, the solutions to this ODE are implicitly given by

$$\psi(t, y) = t^2 - ty + y^2 = C.$$

Incidentally, what does this mean geometrically? Since $\psi(t, y)$ is a quadratic function in two variables, we should expect the solutions to be conic sections (i.e. parabolas, circles, ellipses, or degenerate versions of these such as lines or points), and they should fill out every point in the $(t, y)$ plane and never cross each other. Indeed, we can write

$$t^2 - ty + y^2 = \tfrac{1}{4}(y + t)^2 + \tfrac{3}{4}(y - t)^2$$

(if you've taken a multivariable calculus class with linear algebra you should know how to write any quadratic form in this "diagonal" form, otherwise don't worry about it). This means that most of the solutions are ellipses centered at the origin and rotated by 45 degrees, plus when $C = 0$ we get a point at the origin. Technically these curves aren't globally defined functions $y(t)$ since they don't pass the vertical line test, but they still make sense as solutions if we think of $t$ and $y$ on equal footing.

**Problem 2.** (1) (10 pts / 100) Find the general solution to the ODE

$$y'(t) + y(t) = e^t.$$

Your answer should involve one arbitrary constant.

(2) (10 pts / 100) Find the solution to the initial value problem

$$\begin{cases} y'(t) + e^t y(t) = 1 \\ y(t_0) = y_0. \end{cases}$$

You may leave your answer in the form of a definite integral (e.g. something like $\int_{s=t_0}^{s=t} s^{17} \sin(s) ds$). *Hint: you may encounter the function* $\exp(e^t)$*, which is perfectly fine.*

**Solution 2.** (1) This is a first order linear ODE, so we can solve it using the method of integrating factors. Multiply both sides by $\mu(t)$ for an as yet undetermined function $\mu(t)$. We get

$$y'(t)\mu(t) + y(t)\mu(t) = e^t \mu(t).$$

We want to write the left hand side as $\frac{d}{dt}(y\mu)$ and therefore we want

$$y'\mu + y\mu' = y'\mu + y\mu,$$

i.e. we want

$$\mu' = \mu,$$

so we can take $\mu(t) = e^t$. Then the ODE can be rewitten as

$$\frac{d}{dt}(ye^t) = e^{2t},$$

and integrating both sides with respect to $t$ gives

$$ye^t = (1/2)e^{2t} + C,$$

i.e.

$$y(t) = 1/2e^t + C^{-t}.$$

(2) Similarly, we multiply the ODE by an as yet undetermined function $\mu(t)$ to obtain

$$y'(t)\mu(t) + e^t y(t)\mu(t) = \mu(t),$$

and we seek $\mu(t)$ such that $y'\mu + \mu' y = y'\mu + e^t y\mu$, so we want $\mu' = e^t\mu$. This is a separable ODE, so we can write

$$\frac{d\mu}{\mu} = e^t$$

and integrate both sides to obtain

$$\ln|\mu| = e^t + C.$$

Since we just need a single integrating factor, we can take $\mu(t) = \exp(e^t)$. Therefore the ODE can we written as

$$\frac{d}{dt}(y\exp(e^t)) = \exp(e^t).$$

Integrating both sides from $t_0$ to $t$, applying the fundamental theorem of calculus and using a dummy variable $s$, we obtain

$$y(t)\exp(e^t) - y_0\exp(e^{t_0}) = \int_{s=t_0}^{s=t} \exp(e^s)ds,$$

i.e.

$$y(t) = \exp(-e^t)\left(\int_{s=t_0}^{s=t} \exp(e^s)ds + y_0\exp(e^{t_0})\right)$$

**Problem 3.** (12 pts / 100) Consider the initial value problem

$$\begin{cases} y'(t) = (1 + y(t))^2 \\ y(0) = y_0. \end{cases}$$

Find the solution, and state the maximal interval for $t$ on which the solution is defined. (For example, $(-\infty, \infty)$ would mean the solution is defined for all $t$). Your answer may depend on $y_0$. *Hint: does $y(t)$ have a vertical asymptote? Does it occur for $t < 0$ or $t > 0$? What happens if $y_0 = -1$?*

**Solution 3.** Notice that this ODE says that $y'(t)$ is even greater than $y^2$, and we've seen that the explosion equation $y' = y^2$ reaches a singularity in finite time, so we should expect the same for this ODE. Assuming $y \neq -1$, we can separate variables to write it as

$$\frac{dy}{(1 + y)^2} = dt,$$

and integrate both sides to obtain

$$\frac{-1}{1+y} = t + C.$$

Here we can plug in the initial condition to obtain

$$-1/(1+y_0) = C.$$

Solving for $y(t)$, we obtain

$$y(t) = \frac{1}{\frac{1}{1+y_0} - t} - 1.$$

Now, what about if $y = -1$? We can easily see that $y(t) \equiv -1$ is an equilibrium solution. That means that all other solutions stay either above $y = -1$ or below $y = -1$. For any non-equilibrium solution, a singularity occurs when $t = 1/(1+y_0)$. If $y_0 > -1$, this occurs for positive $t$, and the solution stops being defined at that moment. For $y_0 < -1$, the singularity occurs for negative $t$. Therefore we have that the maximal interval containing $t = 0$ on which the solution is defined is $(-\infty, \infty)$ if $y_0 = -1$, $(-\infty, 1/(1+y_0))$ for $y_0 > -1$, and $(1/(1+y_0), \infty)$ for $y_0 < -1$.

**Problem 4.** (1) (14 pts / 100) Find the general solution to the ODE

$$y''(t) + y'(t) + y(t) = 0.$$

Your answer should be a real-valued function and should involve two arbitrary constants.

(2) (14 pts / 100) Find the general solution to the ODE

$$y''(t) - 2y'(t) + y(t) = 0.$$

Your answer should be a real-valued function and should involve two arbitrary constants.

**Solution 4.** (1) If we make the ansatz $y(t) = e^{rt}$, we can plug this into the ODE and find that $r$ must satisfy the characteristic equation

$$r^2 + r + 1 = 0.$$

This gives roots

$$r = \frac{-1 \pm i\sqrt{3}}{2},$$

and this corresponds to two complex valued solutions

$$e^{-t/2 \pm i\sqrt{3}t/2} = e^{-t/2}(\cos(\sqrt{3}t/2) \pm i\sin(\sqrt{3}t/2)).$$

Using the principle of superposition, we've seen that the real and imaginary parts of either of these solutions give a fundamental set of solutions:

$$y_1(t) = e^{-t/2}\cos(\sqrt{3}t/2)$$
$$y_2(t) = e^{-t/2}\sin(\sqrt{3}t/2).$$

Finally, the general solution is given by

$$y(t) = C_1 e^{-t/2}\cos(\sqrt{3}t/2) + C_2 e^{-t/2}\sin(\sqrt{3}t/2).$$

(2) Similarly, the characteristic equation is $r^2 - 2r + 1$, which has $r = 1$ as a repeated root. In this situation we have a fundamental set of solutions given by

$$y_1(t) = e^t$$
$$y_2(t) = te^t,$$

and the general solution is given by

$$y(t) = C_1 e^t + C_2 te^t.$$

By the way, remember that one way to realize that the second solution should be $te^t$ is to notice that since $Ce^t$ is a solution, perhaps $v(t)e^t$ is a reasonable guess for another solution. The idea is that we try to allow the constant $C$ to vary, giving a function $v(t)$. This method of finding a second solution is sometimes called variation of parameters. If we plug the ansatz $v(t)e^t$ into the ODE, we can easily solve for $v(t)$ and arrive at the general solution.

**Problem 5.** (1) (6 pts / 100) Consider the autonomous ODE given by

$$y'(t) = -y^3 + 7y^2 - 10y = 0.$$

Find the equilibrium points and classify them as stable, unstable, or semistable.

(2) (8 pts / 100) Consider the same ODE as in part (a), now with the initial condition $y(0) = y_0$. For which $y_0$ (if any) does there exist some $t \in (0, \infty)$ such that $y(t) = \frac{1}{2}y_0$?

(3) (6 pts / 100) Give an example of an autonomous ODE of the form $y'(t) = F(y)$ which has unstable equilibria at $y = 1$ and $y = 2$ and a semistable equilibrium at $y = 3$. *Hint: your ODE may also have additional equilibria.*

**Solution 5.** (1) Let's write the ODE as $y'(t) = F(y)$, where

$$F(y) = -y^3 + 7y^2 - 10y = -y(y-2)(y-5).$$

Then the equilibria occur precisely when $F(y) = 0$, i.e. for $y = 0, 2, 5$. To determine what types of equilibria occur it's very helpful to draw the slope field. Basically, since $F(y)$ flips from positive to negative at 0, 0 is a stable equilibrium. Since $F(y)$ flips from negative to positive at 2, 2 is an unstable equilibrium. Since $F(y)$ slips from positive to negative at 5, 5 is a stable equilibrium.

(2) For $y_0 < 0$ the function $y(t)$ is increasing and asymptotically approaches 0, so it will eventually reach $y_0/2$. For $y_0 = 0$ we have $y(t) = 0 = y_0/2$ for all $t$. For $0 < y_0 < 2$, the function $y(t)$ is decreasing and asymptotically approaches 0, so it will eventually reach $y_0/2$. For $2 \le y_0 \le 5$ the function $y(t)$ is nondecreasing, so can never reach $y_0/2$. For $y_0 > 5$, the function $y(t)$ is decreasing and asymptotically approaches 5. Therefore it can reach $y_0/2$ provided $y_0 > 10$. In summary, the condition occurs for $y_0 \in (-\infty, 2) \cup (10, \infty)$.

(3) Basically we can take any function $F(y)$ which vanishes for $y = 1, 2, 3$, flips from negative to positive at 1 and 2, and doesn't change signs at $y = 3$. For example, we could take $F(y) = (y-1)(y-1.5)(y-2)(y-3)^2$.