Final Examination
Math 2030: Ordinary Differential Equations
Columbia University Fall 2018
Instructor: Kyler Siegel

Instructions:

- Please write your answers **in this printed exam**. You may use the back of pages for additional work. You may also use blue books or white paper for scratch work, but these are not to be handed in.
- Solve as many problems of the following problems as you can in the allotted time, which is *two hours and 50 minutes*. I recommend first solving the problems are you most comfortable with before moving on to the more challenging ones. Note that the problems are not ordered by level of difficulty or topic. You do not necessarily need to solve every problem to achieve a good grade.
- All of your ODE solutions should be given in terms of real-valued solutions.
- You do **not** need to justify your answers for full credit (unless explicitly asked to do so), although it may help you get partial credit if some of your final answers are incorrect or incomplete.
- Please turn in all scratch work which is relevant to your submitted answers. Keep in mind that suspected cases of copying or otherwise cheating will be taken very seriously.
- Good luck!!

Name: ____________________

Uni: ____________________

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	10	5	8	8	18	8	6	6	15	10	6	100
Bonus Points:	0	0	0	0	0	0	0	0	0	0	0	0
Score:												

1. *Multiple choice questions. Do not provide any justification. In each case, SELECT ALL THAT APPLY.*

(I) (1 point) Consider the ODE

$$y''(t) + 3t^3 y'(t) + \sin(t)y(t) = \cos(t).$$

Given any two solutions $y_1(t), y_2(t)$, we must have that $3y_1(t) + 5y_2(t)$ is also a solution.
A. True B. False

(II) (1 point) Which of the following is a linear ODE?
A. $y''(t) + \sin(y(t)) = 0$ B. $y'(t) + t^3 \sin(t^3)y(t) = 0$ C. $\sin(t)y''(t) + t^2 y'(t) + e^t y(t) = 0$
D. $y'(t) + e^{y(t)} = 0$

(III) (1 point) Which of the following ODEs has a *unique* solution satisfying $y(0) = 0$, defined at least for $|t|$ sufficiently small? A. $y'(t) = \frac{1}{3}y(t)$ B. $y'(t) = y^{1/3}(t)$ C. $y'(t) = \frac{1}{3}y(t)^2$

(IV) (1 point) Which of the following initial value problems has a solution?
A. $y''(t) + 7y'(t) + 9y(t) = \sin(t)$, $y(0) = 0$, $y'(0) = 4$ B. $y''(t) + 7y'(t) + 9y(t) = \sin(t)$, $y(5) = 0$, $y'(5) = 4$ C. $y'(t) + 6y(t) = 0$, $y(0) = 1$, $y'(0) = 1$ D. $y'(t) + 6y(t) = 0$, $y(0) = 0$, $y'(0) = 0$

(V) (1 point) For $a, b, c, d \in \mathbb{R}$, suppose the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has a repeated eigenvalue. Then there is a unique eigenvector up to scaling.
A. True B. False

(VI) (1 point) For an ODE of the form $y''(t) + p(t)y'(t) + q(t)y(t) = 0$ with $p(t), q(t)$ continuous functions, the Wronskian of two solutions $y_1(t), y_2(t)$ is either never zero or identically zero.

A. True B. False

(VII) (1 point) For an ODE of the form $P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0$ with $P(t), Q(t), R(t)$ continuous functions, the Wronskian of two solutions $y_1(t), y_2(t)$ is either never zero or identically zero.

A. True B. False

(VIII) (1 point) For the ODE

$$t^2(t-1)^2y''(t) + (t-1)y'(t) + \sin(t)y(t),$$

A. $t = 0$ is an ordinary point B. $t = 0$ is a singular point C. $t = 1$ is an ordinary point D. $t = 1$ is a singular point

(IX) (1 point) For the ODE

$$t^k y''(t) + \sin(t)y(t),$$

for which values of k is $t = 0$ an *irregular* singular point?

A. $k = 1$ B. $k = 2$ C. $k = 3$ D. $k = 4$

(X) (1 point) Which of the following ODEs has a solution satisfying $y(0) = 1$ which is defined for all $t \in (-\infty, \infty)$?

A. $y'(t) = 3y(t)$ B. $y'(t) = 3y(t)^2$ C. $y'(t) = 3y(t)^3$ D. $y''(t) + 4y'(t) + 5y(t) = \sin(t)$

2. Consider the system of equations ODEs

$$\begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = -x_1(t) + a x_2(t), \end{cases}$$

where a is a real-valued constant. For each of the following values of a, determine which plot in Figure 1 best represents the solutions.

(I) (1 point) $a = -4$

(II) (1 point) $a = -2$

(III) (1 point) $a = -1$

(IV) (1 point) $a = 0$

(V) (1 point) $a = 1$

3. Consider the ODE

$$ty'(t) - y(t) = t^2 e^{-t}, \qquad t \geq 1.$$

(I) (6 points) Find the solution which satisfies the initial condition $y(1) = y_0$, where y_0 is some real-valued constant. *Note: your answer might depend on* y_0.

(II) (2 points) Find the limit as t approaches $+\infty$ of your solution.

4. Consider the ODE

$$y'(t) = y(t)\sin(y(t)).$$

(I) (5 points) For which values of y is there an equilibrium? Classify each equilibrium as stable, unstable, or semistable.

(II) (3 points) For the solution satisfying the initial condition $y(0) = y_0$, for some $y_0 \in [0, 10]$, find the limit as t approaches $+\infty$. *Note: your answer might depend on* y_0*.*

5. (I) (6 points) Find the general solution to the ODE

$$y^{(4)}(t) + 3y'''(t) + 5y''(t) = 0.$$

(II) (6 points) Find the general solution to the ODE

$$y''(t) + 3y'(t) + 5y(t) = \sin(t).$$

(III) (6 points) Find a particular solution to the ODE

$$y''(t) - 3y'(t) + y(t) = e^{at},$$

where a is some real-valued constant. *Note: your answer might depend on a.*

6. (8 points) Find the general solution to the ODE

$$y(t) + 2ty'(t) - y(t)e^{y(t)}y'(t) = 0.$$

Hint: look for an integrating factor of the form $\mu(x, y) = f(y)$ *for some function* f*. You may use that fact that* $(y^2 - 2y + 2)e^y$ *is an antiderivative for* y^2e^y*.*

7. (6 points) Consider the ODE

$$t^2 y''(t) + 4ty'(t) + 3y(t) = 0, \qquad t > 0.$$

Find a fundamental set of solutions.

8. (6 points) Consider the ODE

$$y''(t) - y(t) = t^{-2}e^{-2t}, \qquad t > 0.$$

Assuming there is a solution of the form $u_1(t)e^t + u_2(t)e^{-t}$, find $u_1'(t)$ and $u_2'(t)$. *Hint: the functions* $u_1(t), u_2(t)$ *are not uniquely determined, and you're welcome to impose additional constraints on them to narrow the search space.*

9. (I) (5 points) Find the general solution to the system of ODEs

$$\begin{cases} x_1'(t) = 2x_1(t) + x_2(t) \\ x_2'(t) = -5x_1(t). \end{cases}$$

(II) (5 points) Find the general solution to the system of ODEs

$$\begin{cases} x_1'(t) = 3x_1(t) - 4x_2(t) \\ x_2'(t) = x_1(t) - x_2(t). \end{cases}$$

(III) (5 points) Find the general solution to the system of ODEs

$$\begin{cases} x_1'(t) = 5x_1(t) \\ x_2'(t) = 7x_2(t) \\ x_3'(t) = x_3(t) + 2x_4(t) \\ x_4'(t) = 2x_3(t) + 4x_4(t) \end{cases}$$

10. (10 points) Consider the ODE

$$y''(t) - 2ty'(t) + 10y(t) = 0.$$

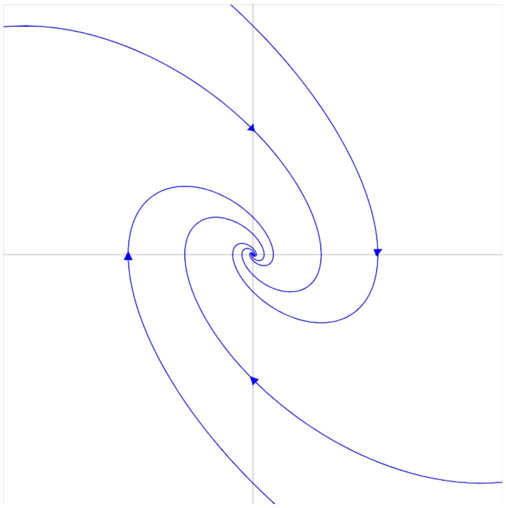
Find a solution which is a nonzero polynomial. *Hint: start with the ansatz that the solution is a power series centered at* $t = 0$.

11. (6 points) Consider the ODE

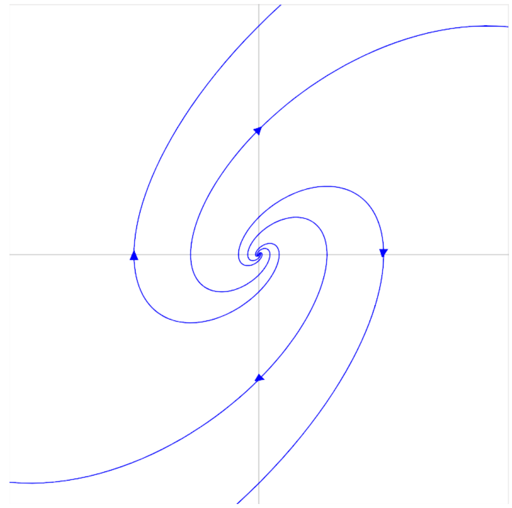
$$t^2 y''(t) + t y'(t) + (t-2)y(t) = 0, \qquad t > 0.$$

Given that there are two linearly independent solutions $y_1(t), y_2(t)$ which are Frobenius series, i.e. of the form $y_1(t) = \sum_{n=0}^{\infty} t^{r_1} a_n t^n$ and $y_2(t) = \sum_{n=0}^{\infty} t^{r_2} b_n t^n$, find r_1 and r_2. *Note: you do not need to find the rest of the coefficients.*

Figure 1



(a)



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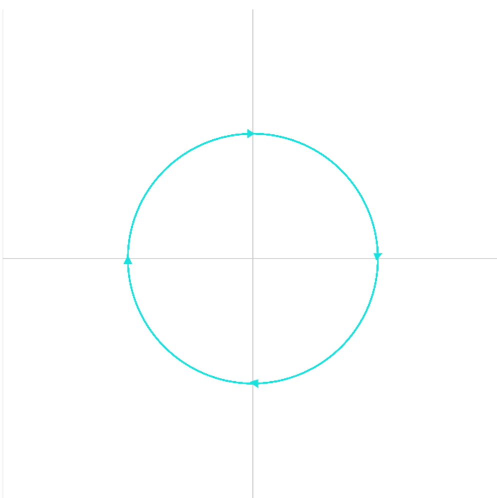
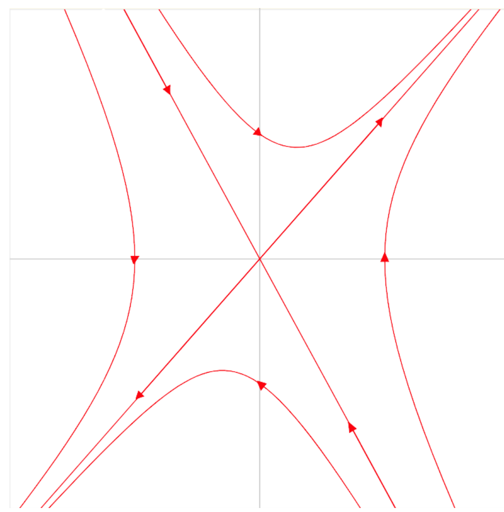
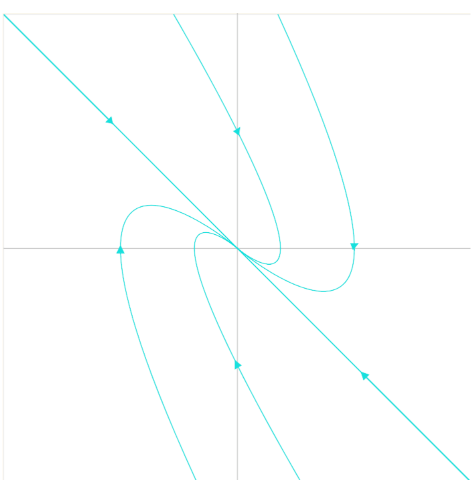


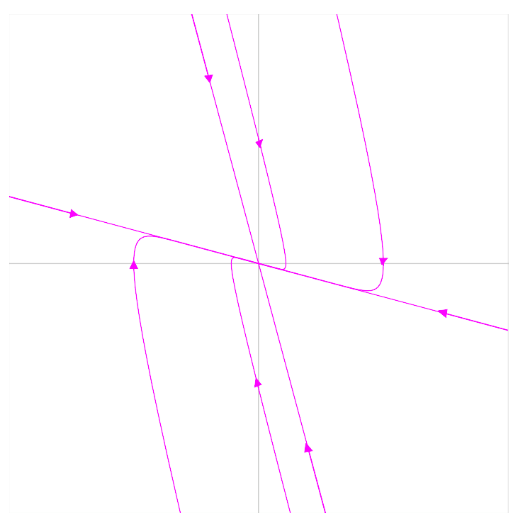
(c)



(d)



(e)



(f)