Section 5.5: 1, 5, 10, 12, 13

1. We have the equation $2xy'' + y' + xy = 0$.

   (a) By dividing through by the coefficient of the second derivative term, we get $p(x) = \frac{1}{2x}$ and $q(x) = \frac{1}{2}$. Then, to check if 0 is a regular singular point, we need to check:

   $$\lim_{x\to 0} xp(x) = \frac{1}{2}, \ \lim_{x\to 0} x^2 q(x) = 0 \ \checkmark$$

   and because both limits are finite, $x = 0$ is a regular singular point.

   (b) We begin with the ansatz $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, so we can plug it into the equation:

   $$2x\sum_{n=0}^{\infty} a_n(n+r-1)(n+r)x^{n+r-2} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$
   $$\sum_{n=0}^{\infty} a_n(n+r)(2n+2r-2+1)x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$$
   $$\sum_{n=0}^{\infty} a_n(n+r)(2n+2r-1)x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$
   $$a_0(r)(2r+1) + \dots + \sum_{n=2}^{\infty} a_n(n+r)(2n+2r-1)x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1} = 0$$

   where the dots refer to the a_1 term from the first sum. Then we know that the coefficients of each x term must be 0, so we can read off the indicial equation from the coefficient of the a_0 term:

   $$\boxed{r(2r+1) = 0} \implies \boxed{r_{1,2} = 0, \frac{1}{2}}$$

   We can also read off the recurrence relation:

   $$\boxed{a_n = -\frac{a_{n-2}}{(n+r)(2n+2r-1)}, n \geq 2}$$

   (c) Then the series solution follows from substitution into the original ansatz and using the larger root:

   $$\boxed{y_1 = x^{1/2}\left[1 - \frac{x^2}{2(5)} + \dots + \frac{(-1)^n x^{2n}}{(2^n)(n!)5(9)(13)\dots(4n+1)} + \dots\right]}$$

   (d) The second solution can be written out the same way:

   $$\boxed{y_2 = 1 - \frac{x^2}{2(3)} + \dots \frac{(-1)^n x^{2n}}{(2^n)(n!)3(7)(11)\dots(4n-1)} + \dots}$$

5. We have $3x^2y'' + 2xy' + x^2y = 0$.

(a) Just as in the previous problem, we can compute the limits of p and q to check if $x = 0$ is a regular singular point:

$$\lim_{x\to 0} \frac{2}{3} = \frac{2}{3}, \lim_{x\to 0} \frac{x^2}{3} = 0\checkmark$$

and because both limits are finite, $x = 0$ is a regular singular point.

(b) Again, we begin with the same ansatz as usual and plug it into the equation:

$$\begin{aligned} 3\sum_{n=0}^{\infty} a_n(n+r-1)(n+r)x^{n+r} + 2\sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_nx^{n+r+2} &= 0 \\ \sum_{n=0}^{\infty} a_n(3n+3r-1)(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_nx^{n+r+2} &= 0 \\ a_0(3r-1)r + a_1 \cdots + \sum_{n=2}^{\infty} \left[a_{n-2} + a_n(n+r)(3n+3r-1)\right] x^{n+r} &= 0 \end{aligned}$$

and we can read off the indicial equation:

$$\boxed{r(3r-1) = 0} \implies \boxed{r_{1,2} = 0, \frac{1}{3}}$$

and the recurrence relation:

$$\boxed{a_n = -\frac{a_{n-2}}{(n+r)(3n+3r-1)}, n \geq 2}$$

(c) We can substitute $r = 1/3$ into the recurrence relation and that into the ansatz to get:

$$\boxed{y_1 = x^{1/3}\left[1 - \frac{x^2}{2(7)} + \cdots + \frac{(-1)^n x^{2n}}{(2^n)(n!)7(13)\dots(6n+1)} + \dots\right]}$$

(d) Substituting $r = 0$, we have

$$\boxed{y_2 = 1 - \frac{x^2}{2(5)} + \dots \frac{(-1)^n x^{2n}}{(2^n)(n!)5(11)(17)\dots(6n-1)} + \dots}$$

10. We have $x^2y'' + (x^2 + \frac{1}{4})y = 0$.

(a) We can follow the same process as before:

$$\lim_{x\to 0} \frac{x^2(x^2+1/4)}{x^2} = \frac{1}{4}\checkmark$$

and $p(x) = 0$ so these limits are finite and $x = 0$ is a regular singular point.

(b) Plugging in the same ansatz, we get:

$$\sum_{n=0}^{\infty} a_n(n+r-1)(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} + \frac{1}{4}\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$
$$a_0\left[(r-1)r + \frac{1}{4}\right] + \cdots + \sum_{n=2}^{\infty}\left[a_n\left((n+r-1)(n+r) + \frac{1}{4}\right) + a_{n-2}\right] = 0$$

so we can read off the indicial equation:

$$\boxed{(r-1)r + \frac{1}{4} = 0} \implies \boxed{r_{1,2} = \frac{1}{2}}$$

and the recurrence relation:

$$\boxed{a_n = -\frac{a_{n-2}}{(n+r-1)(n+r)+\frac{1}{4}}, n \geq 2}$$

(c) Substituting these into the ansatz, we get:

$$\boxed{y_1 = x^{1/2}\left[1 - \frac{x^2}{2} + \cdots + \frac{(-1)^n x^{2n}}{(2^{2n})(n!)} + \dots\right]}$$

12. We now have the equation $(1-x^2)y'' - xy' + \alpha^2 y = 0$.

(a) We can first check that $x = 1$ is a regular singular point by writing $(1-x^2) = -(x-1)(x+1)$:

$$\lim_{x\to 1} \frac{-x(x-1)}{(1-x)(x+1)} = \frac{1}{2}, \lim_{x\to 1} \frac{-\alpha^2(x-1)}{x+1} = 0 \checkmark$$

and for $x = -1$:

$$\lim_{x\to -1} \frac{-x(x+1)}{(1-x)(x+1)} = \frac{1}{2}, \lim_{x\to 1} \frac{\alpha^2(x+1)}{x-1} = 0 \checkmark$$

so both are regular singular points. Now we can make use of the expression for the general form of the indicial equation $F(r) = r(r-1) + p_0 r + q_0 = 0$ where p_0 and q_0 are the limits of p and q evaluated at the singular point, so for $x = 1$:

$$r(r-1) + \frac{3}{2}r = r\left(r - \frac{1}{2}\right) = 0 \implies \boxed{r_{1,2} = 0, \frac{1}{2}}$$

For $x = -1$, we have the same indicial equation, so the same roots.

(b) For this problem, we begin with the ansatz $y = \sum_{n=0}^{\infty} a_n(x-1)^{n+r}$. For substituting this into the ODE, it is useful to rewrite:

$$1 - x^2 = -(x-1)(x+1) = -(x-1)^2 - 2(x-1),\ -x = -(x-1) - 1$$

so we write:

$$-\sum_{n=0}^{\infty}(n+r-1)(n+r)(x-1)^{n+r} - 2\sum_{n=0}^{\infty} a_n(n+r-1)(n+r)(x-1)^{n+r-1}$$
$$-\sum_{n=0}^{\infty} a_n(n+r)(x-1)^{n+r} - \sum_{n=0}^{\infty} a_n(n+r)(x-1)^{n+r-1} + \alpha^2\sum_{n=0}^{\infty} a_n(x-1)^{n+r} = 0$$

Skipping some steps because we are following the same process as the previous few questions and already know the indicial equation, we can read off the coefficient of the x^{n+r} term:

$$a_n = -a_{n-1}\frac{(n+r-2)(n+r-1) + (n+r-1) - \alpha^2}{2(n+r-1)(n+r) + (n+r)} = \boxed{-\frac{(n+r-1)^2 - \alpha^2}{(n+r)(1+2(n+r-1))}}$$

Then we can substitute in the root $r = 1/2$:

$$a_n = -a_{n-1}\frac{\left(\left(\frac{2n-1}{2}\right)^2 - \alpha^2\right)}{(2n+1)n} = -a_{n-1}\frac{((2n-1)^2 - 4\alpha^2)}{(2n+1)4n}$$

so the first solution is:

$$\boxed{y_1 = |x-1|^{1/2}\left[1 - \frac{1-4\alpha^2}{(4)(3)}(x-1) + \cdots + \frac{(-1)^n(1-4\alpha^2)\ldots((2n-1)^2-4\alpha^2)}{2^n(2n+1)!}(x-1)^n\right]}$$

Then if we substitute in $r = 0$:

$$a_n = -a_{n-1}\frac{(n-1)^2 - \alpha^2}{n(2n-1)}$$

and we have another solution:

$$\boxed{y = 1 + \alpha^2(x-1) + \cdots + \frac{(-1)^n(-\alpha^2)(1-\alpha^2)\ldots((n-1)^2-\alpha^2)}{n!(2n-1)!}(x-1)^n}$$

13. We have $xy'' + (1-x)y' + \lambda y = 0$.

(a) Rewriting the equation, we have $y'' + \frac{1-x}{x}y' + \frac{\lambda}{x}y = 0$. Then, we have:

$$p_0 = \lim_{x\to 0}(1-x) = 1,\ q_0 = \lim_{x\to 0}\lambda x^2 = 0\checkmark$$

so $x = 0$ is a singular point.

(b) This corresponds to the indicial equation:

$$\boxed{r(r-1)+r=0} \implies \boxed{r_{1,2}=0}$$

For the recurrence relation, we will substitute in the same ansatz as usual and we have:

$$\sum_{n=0}^{\infty} a_n(n+r-1)(n+r)x^{n+r-1} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} - \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} \lambda a_n x^{n+r} = 0$$

Again skipping steps that are just simple algebra and the same manipulations as before, we have the recurrence relation:

$$\boxed{a_n = \frac{(n+r-1)-\lambda}{(n+r)^2}a_{n-1}}$$

(c) Taking $r=0$, we have a solution of the form:

$$\boxed{y = 1-\lambda x + \frac{(-\lambda)(1-\lambda)}{4}x^2 + \cdots + \frac{(-\lambda)\ldots(n-1-\lambda)}{(n!)^2}x^n + \ldots}$$

From this, it is clear that if $\lambda = m$, for $k > m$, all the a_k coefficients will have a factor of $a_{m+1} = \frac{m+1-1-m}{(m+1)^2} = 0$, so a_{m+1} and all subsequent coefficients will be 0. Therefore, the series truncates and we are left with a polynomial.

Section 5.6: 5, 9

5. We have $x^2y'' + 3\sin xy' - 2y = 0$, which yields $p(x) = \frac{3\sin x}{x^2}$ and $q(x) = \frac{2}{x^2}$. These both have singular points at $x=0$ and we can see that

$$p_0 = \lim_{x\to 0}\frac{3\sin x}{x} = 3,\ q_0 = \lim_{x\to 0} 2 = 2$$

so this is a regular singular point. Then, the indicial equation is

$$\boxed{r(r-1)+3r+2=0} \implies \boxed{r_{1,2} = -1\pm\sqrt{3}}$$

9. We now have $x^2(1-x)y'' - (1+x)y' + 2xy = 0$, which has singular points at $x = 0, 1$. For $x=0$, we see that

$$\lim_{x\to 0}\frac{-(1+x)}{x(1-x)} \to \text{DNE}$$

so this is an irregular singular point. For $x = 1$, we find that

$$p_0 = \lim_{x \to 1} \frac{1+x}{x^2} = 2, \; q_0 = \lim_{x \to 1} \frac{-2(x-1)}{x} = 0$$

Therefore, $x = 1$ is a regular singular point. Then we find that the indicial equation

$$\boxed{r(r-1) + 2r = 0} \implies \boxed{r_{1,2} = 0, -1}$$