Section 4.1: 9, 10, 15

9. We have $f_1(t) = 2t - 3$, $f_2(t) = t^2 + 1$, $f_3(t) = 2t^2 - t$, and $f_4(t) = t^2 + t + 1$. If these functions are linearly dependent, then there exists a set of integers $k_1, k_2, k_3, k_4 \in \mathbb{Z}$ such that

$$0 = k_1 f_1 + k_2 f_2 + k_3 f_3 + k_4 f_4 = k_1(2t - 3) + k_2(t^2 + 1) + k_3(2t^2 - t) + k_4(t^2 + t + 1)$$

We can group terms and observe that $-3k_1 + k_2 + k_4 + (2k_1 - k_3 + k_4)t + (k_2 + 2k_3 + k_4)t^2 = 0$. For this expression to hold for all t, we must have all the coefficients be 0:

$$\begin{aligned} -3k_1 + k_2 + k_4 &= 0 \\ 2k_1 - k_3 + k_4 &= 0 \\ k_2 + 2k_3 + k_4 &= 0 \end{aligned}$$

This is a system of three linear equations with four unknowns, so it is underconstrained and we must have one free parameter. In particular, this means that the given functions are **linearly dependent**. We are asked to find a linear relation among them, so we can solve the system and find that $k_1 = -\frac{2}{7}k_4$, $k_2 = -\frac{13}{7}k_4$, $k_3 = \frac{3}{7}k_4$, and k_4 is a free parameter. Taking $k_4 = 1$ gives one possible solution:

$$\boxed{k_1 = -\frac{2}{7},\ k_2 = -\frac{13}{7},\ k_3 = \frac{3}{7},\ k_4 = 1}$$

10. This problem is near identical to the previous one, with a slight modification to f_2. Following the same procedure, we find that it yields four equations for our four parameters, indicating that the system is likely linearly independent. However, we can also see this from the Wronskian of the system of functions:

$$W(f_1, f_2, f_3, f_4) = \begin{vmatrix} 2t-3 & t^3+1 & 2t^2-t & t^2+t+1 \\ 2 & 3t^2 & 4t-1 & 2t+1 \\ 0 & 6t & 4 & 2 \\ 0 & 6 & 0 & 0 \end{vmatrix} = 156 \neq 0$$

Therefore, this is a **linearly independent** system of functions.

15. We are given the ODE $xy''' - y'' = 0$ and the solutions $y_1 = 1$, $y_2 = x$, $y_3 = x^3$. Verifying that they satisfy the equation is simple:

$$\begin{gathered} 1\ \ y''' = y'' = 0 \text{ so } x(0) - 0 = 0\ \checkmark \\ 2\ \ y''' = y'' = 0 \text{ so } x(0) - 0 = 0\ \checkmark \\ 3\ \ y'' = 6x,\ y''' = 6, \text{ so we have } x(6) - 6x = 0\ \checkmark \end{gathered}$$

Then we can compute the Wronskian:

$$W(y_1, y_2, y_3) = \begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} = \boxed{6x}$$

Section 4.2: 1, 2, 5, 8, 10, 11, 14, 15, 21

1. We want to write $z = 1+i = Re^{i\theta}$. If $z = a+bi$, then $R = |z| = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$ and $\theta = \arctan(b/a) = \pi/4 + 2\pi n$, where $n \in \mathbb{Z}$ because tan is 2π periodic, so we have:
$$z = 1 + i = \boxed{\sqrt{2}e^{i\left(\frac{\pi}{4}+2\pi n\right)},\ n \in \mathbb{Z}}$$

2. Following the same procedure, we find that $R = 2$ and $\theta = 2\pi/3 + 2\pi n$, so we have:
$$z = -1 + i\sqrt{3} = \boxed{2e^{i\left(\frac{2\pi}{3}+2\pi n\right)},\ n \in \mathbb{Z}}$$

5. This is the same procedure again:
$$z = 1 - i\sqrt{3} = \boxed{2e^{i\left(\frac{11\pi}{6}+2\pi n\right)},\ n \in \mathbb{Z}}$$

8. We want to find the square roots of $1-i$. To do this, we first express $1-i$ in polar form following the method of the previous three questions: $1 - i = \sqrt{2}e^{i\left(-\frac{pi}{4}+2\pi n\right)}$. Then we can take the root:
$$\left(\sqrt{2}e^{i\left(-\frac{\pi}{4}+2\pi n\right)}\right)^{1/2} = 2^{1/4}e^{i\left(-\frac{\pi}{8}+\pi n\right)},\ n \in \{0,1\}$$
Our two roots are then
$$(1-i)^{1/2} = \boxed{2^{1/4}\left(\cos\left(\frac{\pi}{8}\right) - i\sin\left(\frac{\pi}{8}\right)\right),\ 2^{1/4}\left(\cos\left(\frac{7\pi}{8}\right) + i\sin\left(\frac{7\pi}{8}\right)\right)}$$

10. We can follow the same procedure as the previous question
$$\left(2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)\right)^{1/2} = \left(2e^{i\frac{\pi}{3}}\right)^{1/2} = \sqrt{2}\left(\cos\left(\frac{\pi}{6}+n\pi\right) + i\sin\left(\frac{\pi}{6}+n\pi\right)\right)$$
$$= \boxed{\sqrt{\frac{3}{2}} + i\frac{1}{\sqrt{2}},\ -\sqrt{\frac{3}{2}} - i\frac{1}{\sqrt{2}}}$$

11. We have the equation $y''' - y'' - y' + y = 0$, and assuming a general form of the solution $y = e^{rt}$, we get the characteristic equation $r^3 - r^2 - r + 1 = (r^2-1)(r-1) = 0$, which has roots $r = \pm 1$, where the root $+1$ has multiplicity 2. This yields the general solution:
$$\boxed{y = c_1e^t + c_2te^t + c_3e^{-t}}$$

14. The equation $y^{(4)} - 4y''' + 4y'' = 0$ yields the characteristic equation $r^4 - 4r^3 + 4r^2 = r^2(r-2)^2 = 0$ which has roots $r = 0, 2$ both of which have multiplicity 2. The general solution is:
$$\boxed{y = c_1 + c_2t + c_3e^{2t} + c_4te^{2t}}$$

15. The equation $y^{(6)} + y = 0$ has the characteristic equation $r^6 + 1 = 0$. This is less straightforward to solve, but we can recall the method from problems 8 and 10 from this section to find the roots. Namely, expressing $-1 = e^{i(\pi+2\pi n)}$ in polar form, we have

$$r = \left(e^{i(\pi+2\pi n)}\right)^{1/6} = e^{i\left(\frac{\pi}{6}+\frac{\pi n}{3}\right)} = \frac{\sqrt{3}}{2} \pm \frac{i}{2},\ -\frac{\sqrt{3}}{2} \pm \frac{i}{2},\ \pm i$$

Then the general solution is a superposition of the solutions that correspond to these roots and noting that for a root that takes the form $r = \lambda + i\mu$, $y = ce^{rt} = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)$. Then our general solution is

$$\boxed{y = e^{\frac{\sqrt{3}}{2}t}\left(c_1\cos\frac{t}{2} + c_2\sin\frac{t}{2}\right) + e^{-\frac{\sqrt{3}}{2}t}\left(c_3\cos\frac{t}{2} + c_4\sin\frac{t}{2}\right) + c_5\cos(t) + c_6\sin(t)}$$

21. The equation $y^{(8)} + 8y^{(4)} + 16y = 0$ yields the characteristic equation $r^8 + 8r^4 + 16 = (r^4+4)^2 = 0$, which has four roots of multiplicity 2:

$$r = 4^{1/4}e^{i\left(\frac{\pi}{4}+\frac{\pi n}{2}\right)} = \sqrt{2}\left(\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}\right), \sqrt{2}\left(-\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}\right) = 1 \pm i, -1 \pm i$$

Then we can read off the general form of the solution in the same way as the previous problem (noting that we have repeated roots):

$$\boxed{y = e^t\left[(c_1 + c_2 t)\cos t + (c_3 + c_4 t)\sin t\right] + e^{-t}\left[(c_5 + c_6 t)\cos t + (c_7 + c_8 t)\sin t\right]}$$

Section 4.3: 1, 2, 6

1. We have the differential equation $y''' - y'' - y' + y = 2e^{-t} + 3$. Looking to the homogeneous part first, we can recognize that this is the same problem as 4.2, question 11, and just cite that result. As such, we have $y_H = c_1 e^t + c_2 te^t + c_3 e^{-t}$. For the particular solution, we can split up the right hand side of the equation into the two terms and solve for the coefficients as independent equations. Namely, we have $y''' - y'' - y' + y = 2e^{-t}$ and $y''' - y'' - y' + y = 3$. The first equation points to a solution of the form $y_{p,1} = c_4 te^{-t}$, where we have a factor of t multiplying the exponential because -1 was a root of our characteristic equation. Then we can plug this into the ODE, compute the derivatives, and solve for c_4:

$$e^{-t}(3c_4 + 2c_4 - c_4) + te^{-t}(-c_4 - c_4 + c_4 + c_4) = 2e^{-t} \implies 3c_4 + 2c_4 - c_4 = 2 \implies c_4 = \frac{1}{2}$$

We can solve for the second part of the particular solution easily because it is just a constant and we get $y_{p,2} = 3$. Then our general solution for the ODE is:

$$y = y_H + y_{p,1} + y_{p,2} = \boxed{c_1 e^t + c_2 te^t + c_3 e^{-t} + \frac{1}{2}te^{-t} + 3}$$

2. This problem requires the same approach as the previous, but unfortunately, we did not already solve for the homogeneous solution, so we can't just cite the result. Luckily, the characteristic equation is simple $r^4-1=(r^2-1)(r^2+1)=(r+1)(r-1)(r+i)(r-i)=0$, so we have roots $r=\pm 1,\pm i$. Then our homogeneous solution is $y_H=c_1e^{-t}+c_2e^t+c_3\cos t+c_4\sin t$. As before, we can split the inhomogeneous part into two terms and solve them independently. The first term is again simple, and assuming the form $y_{p,1}=c_5t$, we find that $c_5=-3$. The second part follows after a little bit of algebra, assuming $y_{p,2}=c_6t\cos t+c_7t\cos t$ (because i is a root of our characteristic equation):

$$4c_6\sin t-4c_7\cos t=\cos t \implies c_6=0,\ -4c_7=1 \implies c_7=-\frac{1}{4}$$

This means our general solution is:

$$y=y_H+y_{p,1}+y_{p,2}=\boxed{c_1e^{-t}+c_2e^t+c_3\cos t+c_4\sin t-3t-\frac{1}{4}t\sin t}$$

6. We have $y^{(4)}+2y''+y=3+\cos 2t$. Solving the homogeneous equation gives the characteristic polynomial $r^4+2r^2+1=(r+i)^2(r-i)^2=0$, so we have $r=\pm i$ with multiplicity 2. Then the homogenous solution is $y_H=c_1\cos t+c_2\sin t+c_3t\cos t+c_4t\sin t$. Solving the particular equation, the first term is again simple and is $y_{p,1}=3$. The second term requires a bit of algebra again, assuming $y_{p,2}=c_6\cos 2t+c_7\sin 2t$. We find that $16c_7-8c_7+c_7=0$, so $c_7=0$ and $16c_6-8c_6+c_6=1$, so $c_6=1/9$. Then the general solution is:

$$y=y_H+y_{p,1}+y_{p,2}=\boxed{c_1\cos t+c_2\sin t+c_3t\cos t+c_4t\sin t+3+\frac{1}{9}\cos 2t}$$