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Math 2030
Problem Set 1 Solutions

1.2

7. Field mouse population $\frac{dp}{dt} = 0.5p - 450$

Solve First order separable DE by integral: $\int_{P_0}^{P} \frac{dp}{p-900} = \int_0^t \frac{1}{2} dt$ — P → population at time t; P_0 → Population at time $t=0$

Integral yields $\ln\left(\frac{P-900}{P_0-900}\right) = \frac{t}{2}$

Algebraic manipulation yields $\boxed{P(t) = (P_0 - 900)e^{\frac{t}{2}} + 900}$ as solution to DE

a) Solve for t when mice are extinct, i.e. $P(t) = 0$ when $P_0 = 850$

$0 = -50e^{\frac{t}{2}} + 900$

$\frac{-900}{-50} = e^{\frac{t}{2}} \Rightarrow \boxed{t_{ex} = 2\ln(18)}$ by Algebra

b) Find time of extinction for $0 < P_0 < 900$

Algebraically manipulate our solution $0 = (P_0 - 900)e^{\frac{t}{2}} + 900$

$\hookrightarrow \frac{-900}{P_0 - 900} = e^{\frac{t}{2}} \Rightarrow \boxed{t_{ex} = 2\ln\left(\frac{-900}{P_0 - 900}\right)}$

c) Find P_0 if population becomes extinct in 1 year (i.e. $t = 12$ for 12 months)

$0 = (P_0 - 900)e^{\frac{12}{2}} + 900$

By algebraic manipulation $\boxed{P_0 = 900(1 - e^{-\frac{12}{2}})}$

8. Field mouse population where $\frac{dp}{dt} = rp$

Solve separable DE by integration: $\int_{P_0}^{P} \frac{dp}{P} = \int_0^t r\,dt$

Integral yields $\ln\left(\frac{P}{P_0}\right) = rt$, so by algebraic manipulation $p = P_0 e^{rt}$

a) Find rate r if population doubles in 30 days (i.e. $p = 2P_0$)

Thus $2 = e^{r\,30\text{days}} \rightarrow \ln 2 = 30\text{days} \cdot r$

by algebraic manipulation $\boxed{r = \frac{\ln 2}{30}\,\text{day}^{-1}}$

b) Find r if population doubles in N days i.e. $2 = e^{rN\text{days}}$

by same algebra $\boxed{r = \frac{\ln 2}{N}\,\text{day}^{-1}}$ (units are important)

Classify DE

1.3 1. $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 2y = \sin t$: 2nd order linear DE as there is a second order derivative and no nonlinear y term

2. $(1+y^2)\frac{d^2y}{dt} + t\frac{dy}{dt} + y = e^t$: 2nd order nonlinear DE because there is 2ndorder deriv. and a nonlinear y term: $(1+y^2)\frac{d^2y}{dt}$

3. $\frac{d^4y}{dt^4} + \frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 1$: 4th order linear DE because 4th order deriv. termand no nonlinear y

4. $\frac{dy}{dt} + ty^2 = 0$: 1storder nonlinear DE w/ 1st order deriv. and nonlinear ty^2 term

5. $\frac{d^2y}{dt^2} + \sin(t+y) = \sin t$: 2nd order nonlinear DE w/ 2nd order derivative and $\sin(t+y)$ nonlinear term

6. $\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (\cos^2 t)y = 0$: 2nd order linear DE because of 2nd order derivative and all linear y terms

14. Verify that the given function is solution to DE

$y' - 2ty = 1$ $\quad y = e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}$

First take time derivative of y

$$y' = 2te^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}\frac{d}{dt}\int_0^t e^{-s^2}\,ds + 2te^{t^2} = 2te^{t^2}\int_0^t e^{-s^2}\,ds + 1 - \cancel{e^{t^2}} + 2te^{t^2}$$

Evaluate by FTC $\int_0^t \frac{d[e^{-s^2}]}{ds}\,ds = e^{-t^2} - 1$

Plug in:

$$\cancel{2te^{t^2}\int_0^t e^{-s^2}\,ds} + \cancel{2te^{t^2}} + 1 - \cancel{2te^{t^2}\int_0^t e^{-s^2}\,ds} - \cancel{2te^{t^2}} = 1 \checkmark$$

Algebra works

20. Determine values of r for which DE has solutions $y=t^r$, $t>0$

$t^2y'' - 4ty' + 4y = 0$

Start by assuming $y=t^r$

Take derivatives: $y' = rt^{r-1}$, $y'' = (r-1)(r)t^{r-2}$

Plug in: $t^2(r-1)rt^{r-2} - 4trt^{r-1} + 4t^r = 0$

$$(r-1)rt^r - 4rt^r + 4t^r = 0$$

$$(r-1)r - 4r + 4 = 0$$

$$r^2 - 5r + 4 = 0$$

$$(r-4)(r-1) = 0$$

$$r=4, \quad r=1$$

So for values of $r=4$ and $r=1$, DE has solution of for $y=t^r$

2.2 1. Solve DEs $y' = \dfrac{x^2}{y}$

Separable: $\int_{y_0}^{y} y\,dy = \int_0^x x^2\,dx$ Solve by integration

$$\frac{y^2 - y_0^2}{2} = \frac{x^3}{3} \Rightarrow \frac{y^2}{2} = \frac{x^3}{3} + \frac{y_0^2}{2} \quad \text{OR} \quad \boxed{3y^2 + 2x^3 = C}$$

2. Solve $y' = \dfrac{x^2}{y(1+x^3)}$

Solve separable DE by integration $\int y\,dy = \int \frac{x^2}{1+x^3}\,dx$

$u = 1+x^3$, $du = 3x^2 dx$

$$\int \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|1+x^3|$$

$$\int y\,dy = \frac{y^2}{2} \therefore \frac{y^2}{2} = \frac{1}{3}\ln|1+x^3| + C \Rightarrow \boxed{3y^2 - 2\ln|1+x^3| = C}$$

3. Solve $y' + y^2\sin x = 0$

Manipulate into separable form: $\dfrac{dy}{dx} = -y^2\sin x$

Solve by integration $-\int \frac{dy}{y^2} = \int \sin x\,dx \Rightarrow \frac{1}{y} = -\cos x + C \Rightarrow \boxed{y^{-1} + \cos x = C}$

5. Solve $y' = (\cos^2 x)(\cos^2 2y)$

Get into separableform: $\int \frac{dy}{\cos^2 2y} = \int \cos^2 x \, dy$

$\int \sec^2 2y \, dy = \int \cos^2 x \, dx$

recognize

$\int \sec^2(2y)\, dy = \frac{\tan(2y)}{2} + C$

$\int \cos^2 x \, dx = \int \underbrace{\frac{1+\cos 2x}{2}}_{\text{trig sub.}} dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C$

$\therefore \boxed{\frac{\tan(2y)}{2} - \frac{x}{2} - \frac{\sin(2x)}{4} = C}$

7. $\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^{y}} \Rightarrow$ Use separability of DE to evaluate by integral:

$\int y + e^{y} \, dy = \int x - e^{-x} dx$

$\frac{y^2}{2} + e^{y} = \frac{x^2}{2} + e^{-x} + C$

Solution: $\boxed{\frac{y^2}{2} + e^{y} - \frac{x^2}{2} - e^{-x} = C}$

8. $\frac{dy}{dx} = \frac{x^2}{1+y^2} \rightarrow$ Again use separability of DE:

$\int dy \, (1+y^2) = \int x^2 dx$

$y + \frac{y^3}{3} = \frac{x^3}{3} + C$ by evaluating integral

$\therefore$ Solution: $\boxed{y + \frac{y^3}{3} - \frac{x^3}{3} = C}$