MATH 2030 ODE: Problem Set 11 Solutions

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List of Problems:

- §7.5: 1,11,12,16
- §7.6: 1,9
- §7.8: 1,5

* Python codes were used to create direction field and trajectories of the system. The codes are included in the appendix section at the end.

Chapter 7.5

Problem 1

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{x}$$

(a)

The first step that you take in solving a system of linear equations is to find the eigenvalues and eigenvectors of the matrix.

Let r and ξ be the eigenvalue and eigenvector of the matrix:

$$\begin{aligned} \begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix} &= (3-r)(-2-r)+4 \\ &= r^2 - r - 6 + 4 \\ &= (r-2)(r+1) = 0 \\ \Rightarrow r_1 = -1, r_2 = 2 & \end{aligned}$$

For $r_1 = -1$, the eigenvector can be determined by:

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $r_2 = 2$, the eigenvector can be determined by:

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

The fundamental set of solutions of the equation is:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

Therefore, the general solution to this problem is:

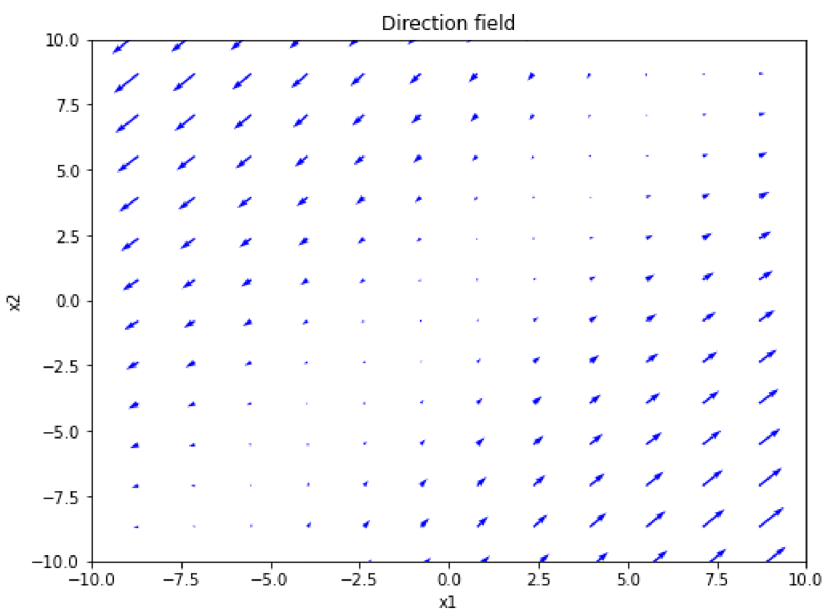
$$\boxed{\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}}$$

where c_1, c_2 are constants.

$$\lim_{t\to\infty} e^{-t} = 0, \quad \lim_{t\to\infty} e^{2t} = \infty$$

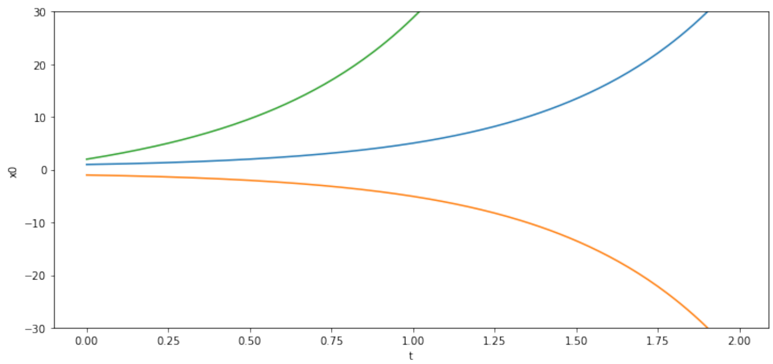
Therefore, as $t \to \infty$, the solution approaches zero when $c_2 = 0$, and the solution approaches infinite when $c_2 \neq 0$.

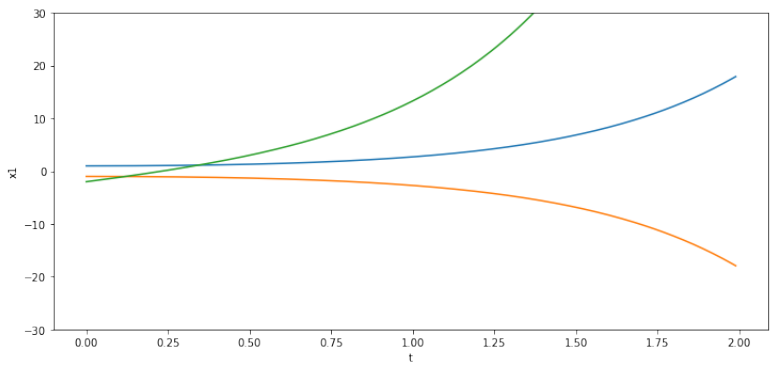
(b)



This plot shows the direction field for the system.

The graph shows the plot with initial conditions $(1,1), (-1,-1), (2,-2)$





Problem 11

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \mathbf{x}$$

Find the eigenvalue and eigenvector of this matrix:

$$\begin{vmatrix} 1-r & 1 & 2 \\ 1 & 2-r & 1 \\ 2 & 1 & 1-r \end{vmatrix} = (1-r)\begin{vmatrix} 2-r & 1 \\ 1 & 1-r \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 2 & 1-r \end{vmatrix} + 2\begin{vmatrix} 1 & 2-r \\ 2 & 1 \end{vmatrix}$$
$$= (1-r)[(1-r)(2-r)-1] - [(1-r)-2] + 2[1-2(2-r)]$$
$$= -(r-1)(r+1)(r-4) = 0$$

$\Rightarrow r_1 = 1, r_2 = -1, r_3 = 4$

For $r_1 = 1$, the eigenvector can be determined by:

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

For $r_2 = -1$, the eigenvector can be determined by:

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $r_3 = 4$, the eigenvector can be determined by:

$$\begin{pmatrix} -3 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \boldsymbol{\xi}^{(3)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The fundamental set of solutions of the equation is:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}, \qquad \mathbf{x}^{(3)}(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t}$$

Therefore, the general solution to this problem is:

$$\boxed{\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{4t}}$$

where c_1, c_2, c_3 are constants.

Problem 12

$$\mathbf{x}' = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \mathbf{x}$$

Find the eigenvalue and eigenvector of this matrix:

$$\begin{aligned} \begin{vmatrix} 3-r & 2 & 4 \\ 2 & -r & 2 \\ 4 & 2 & 3-r \end{vmatrix} &= (3-r)\begin{vmatrix} -r & 2 \\ 2 & 3-r \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-r \end{vmatrix} + 4\begin{vmatrix} 2 & -r \\ 4 & 2 \end{vmatrix} \\ &= (3-r)[-r(3-r)-4] - 2[2(3-r)-8] + 4[4+4r] \\ &= -(r+1)^2(r-8) = 0 \end{aligned}$$

$\Rightarrow r_1 = r_2 = -1, r_3 = 8$

For $r_1 = r_2 = -1$, the eigenvector can be determined by:

$$\begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}, \quad \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

For $r_3 = 8$, the eigenvector can be determined by:

$$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \boldsymbol{\xi}^{(3)} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

The fundamental set of solutions of the equation is:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} e^{-t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}, \qquad \mathbf{x}^{(3)}(t) = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t}$$

Therefore, the general solution to this problem is:

$$\boxed{\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} e^{8t}}$$

where c_1, c_2, c_3 are constants.

Problem 16

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

At first find the general solution of the system of equations. Then, find the exact value of the constants from the given initial conditions.

Determine the eigenvalues and eigenvectors of the matrix:

$$\begin{aligned} \begin{vmatrix} -2-r & 1 \\ -5 & 4-r \end{vmatrix} &= (-2-r)(4-r) + 5 \\ &= (r-3)(r+1) = 0 \\ \Rightarrow r_1 = 3, r_2 = -1 & \end{aligned}$$

When $r_1 = 3$, the eigenvector will be:

$$\begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

When $r_2 = -1$, the eigenvector will be:

$$\begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore, the general solution will be:

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

From the initial condition, we have:

$$c_1 \begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

By solving this system, we get:

$$c_1 = \frac{1}{2}, c_2 = \frac{1}{2}$$

The final solution is:

$$\boxed{\mathbf{x} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}}$$

$$\lim_{t\to\infty} e^{-t} = 0, \quad \lim_{t\to\infty} e^{3t} = \infty$$

Since the terms in front of e^{3t} is positive, the solution approaches infinite as $t \to \infty$.

Chapter 7.6

Problem 1

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}$$

(a)

Find the eigenvalue and eigenvector of the matrix:

$$\begin{vmatrix} 3-r & -2 \\ 4 & -1-r \end{vmatrix} = (3-r)(-1-r) + 8$$
$$= r^2 - 2r + 5 = 0$$

Use the quadratic formula to find the eigenvalue:

$$r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

Find the eigenvector of $r_1 = 1 + 2i$

$$\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

Hint: To find a complex eigenvector, it might be a good idea to set one variable to be a real number, such as 1 or 2.

To solve this problem, we would use Euler's formula, which is $e^{ix} = \cos x + i \sin x$.

$$\begin{aligned}\boldsymbol{\xi}^{(1)}e^{(1+2i)t} &= e^t\begin{pmatrix}1\\1-i\end{pmatrix}(\cos 2t + i\sin 2t)\\ &= e^t\begin{pmatrix}\cos 2t\\ \cos 2t+\sin 2t\end{pmatrix} + ie^t\begin{pmatrix}\sin 2t\\ \sin 2t-\cos 2t\end{pmatrix}\end{aligned}$$

Since we don't want to have complex value in our solution, we can adjust the constants to let the final solution in terms of real-valued functions.

The final solution is:

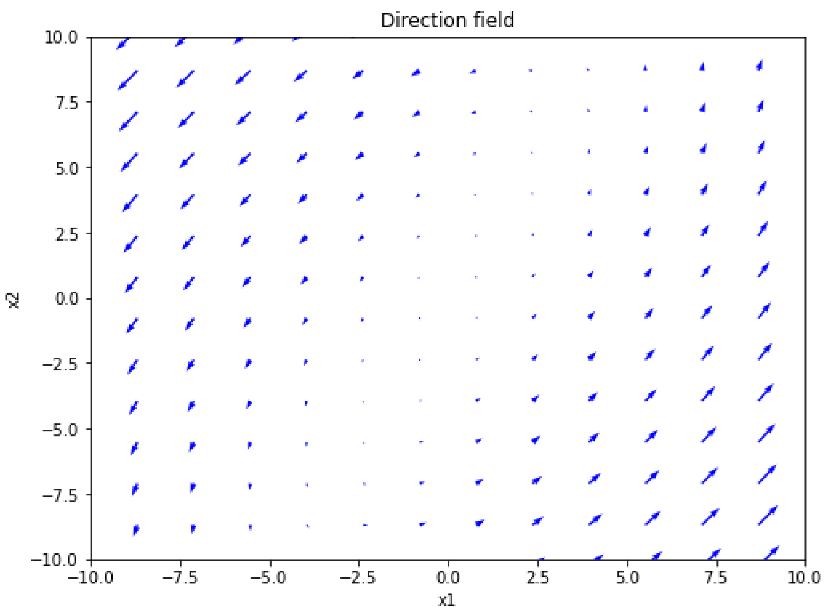
$$\boxed{\mathbf{x} = c_1e^t\begin{pmatrix}\cos 2t\\ \cos 2t+\sin 2t\end{pmatrix} + c_2e^t\begin{pmatrix}\sin 2t\\ \sin 2t-\cos 2t\end{pmatrix}}$$

Here, c_1, c_2 are constants like usual.

(b)

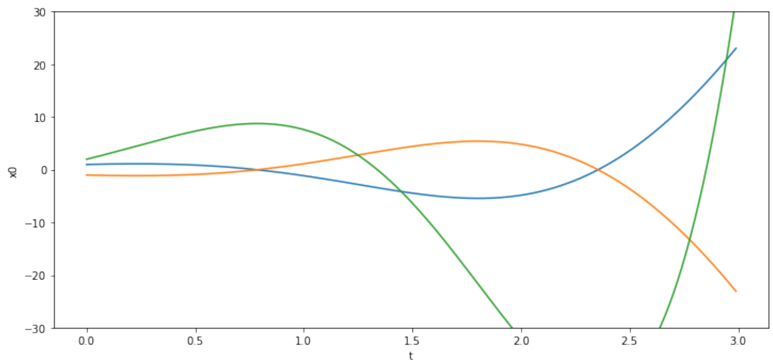
$$\lim_{t\to} e^t = \infty$$

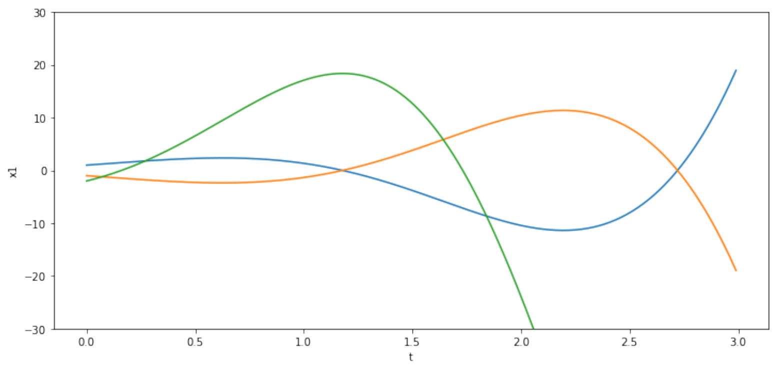
and $\cos 2t, \sin 2t$ don't approach to zero as $t \to \infty$, so the solution approaches infinite as $t \to \infty$.



This plot shows the direction field for the system

The graph shows the plot with initial conditions $(1,1), (-1,-1), (2,-2)$





Problem 9

$$\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

At first, find the general solution of the given system of equations. Then, find the constant of the solution from the given initial conditions.

$$\begin{vmatrix} 1-r & -5 \\ 1 & -3-r \end{vmatrix} = (1-r)(-3-r) + 5$$
$$= r^2 + 2r + 2 = 0$$

Use the quadratic formula to find the roots of the characteristic equation.

$$r = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

Find the eigenvector when $r = -1 + i$.

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$

$$\boldsymbol{\xi}^{(1)}e^{(-1+i)t} = e^{-t}\begin{pmatrix}2+i\\1\end{pmatrix}(\cos t + i\sin t)$$
$$= e^{-t}\begin{pmatrix}2\cos t - \sin t\\ \cos t\end{pmatrix} + ie^{-t}\begin{pmatrix}\cos t + 2\sin t\\ \sin t\end{pmatrix}$$

Since we don't want to have complex value in our solution, we can adjust the constants to let the final solution in terms of real-valued functions.

The general solution is:

$$\mathbf{x} = c_1e^{-t}\begin{pmatrix}2\cos t - \sin t\\ \cos t\end{pmatrix} + c_2e^{-t}\begin{pmatrix}\cos t + 2\sin t\\ \sin t\end{pmatrix}$$

At $t = 0$, $e^{-t} = 1, \cos t = 1, \sin t = 0$. Substitute $t = 0$ to the general solution and find the constants depending on the initial condition:

$$c_1\begin{pmatrix}2\\1\end{pmatrix} + c_2\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \quad c_1 = 1, \quad c_2 = -1$$

Substitute this to the general formula found before:

$$\mathbf{x} = e^{-t}\begin{pmatrix}2\cos t - \sin t\\ \cos t\end{pmatrix} - e^{-t}\begin{pmatrix}\cos t + 2\sin t\\ \sin t\end{pmatrix}$$
$$\boxed{\mathbf{x} = e^{-t}\begin{pmatrix}\cos t - 3\sin t\\ \cos t - 3\sin t\end{pmatrix}}$$

$$\lim_{t\to\infty} e^{-t} = 0, \quad \lim_{t\to\infty} \cos t \leq 1, \quad \lim_{t\to\infty} \sin t \leq 1$$

Therefore, the solution approaches zero as $t \to \infty$.

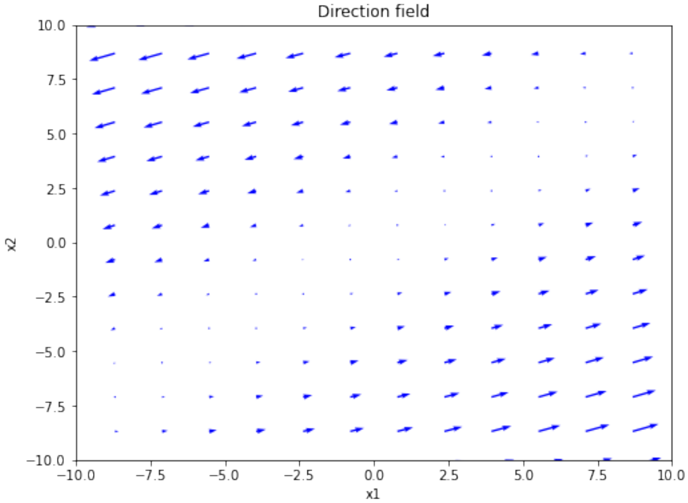
Chapter 7.8

Problem 1

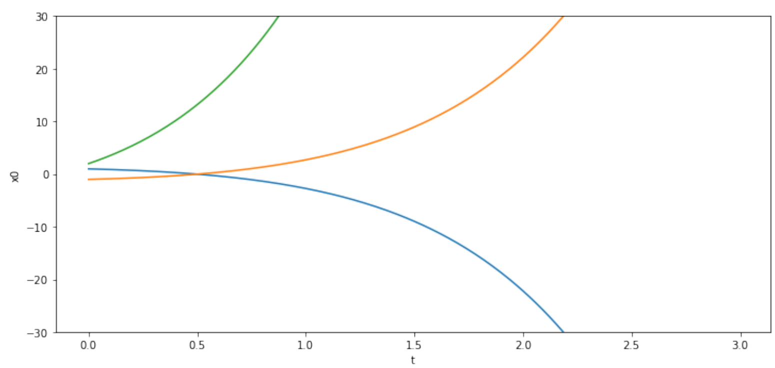
$$\mathbf{x}' = \begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}\mathbf{x}$$

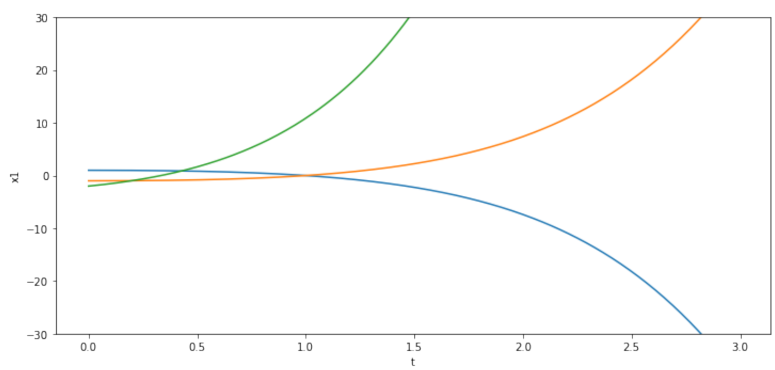
(a)

This plot shows the direction field for the system



The graph shows the plot with initial conditions $(1, 1), (-1, -1), (2, -2)$





(b)

The direction field shows that the solution approaches infinite as $t \to \infty$. We will also be using the analytic solution determined from (c) to verify this fact.

(c)

As in the previous section, find the eigenvalue of the matrix:

$$\begin{aligned}\begin{vmatrix} 3-r & -4 \\ 1 & -1-r \end{vmatrix} &= (3-r)(-1-r)+4 \\ &= (r-1)^2 = 0 \qquad \Rightarrow \qquad r_1 = r_2 = 1\end{aligned}$$

Find the eigenvector corresponding to $r = 1$:

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Here, we are able to obtain 1 solution, but we need a second linearly independent solution to find a fundamental set of solutions to this problem.

Substitute $\mathbf{x} = \boldsymbol{\xi} te^t + \boldsymbol{\eta} e^t$ to the problem. Here, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are constant vectors.

$$\begin{aligned}\mathbf{x}' &= \boldsymbol{\xi} te^t + (\boldsymbol{\xi} + \boldsymbol{\eta})e^t \\ &= \mathbf{A}(\boldsymbol{\xi} te^t + \boldsymbol{\eta} e^t)\end{aligned}$$

Here, $\mathbf{A}$ is the original matrix that is given by the problem. By comparing the coefficients on $\boldsymbol{\xi} te^t$ and $\boldsymbol{\eta} e^t$, we get the following equations:

$$(\mathbf{A} - \mathbf{I})\boldsymbol{\xi} = \mathbf{0} \qquad (\mathbf{A} - \mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$$

By observing these equations, $\boldsymbol{\xi}$ is the eigenvector of $\mathbf{A}$, which was defined from the previous part. So we have the following equation:

$$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Thus, we have: $\eta_1 - 2\eta_2 = 1$

This implies that:

$$\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The general form of the solution will be:

$$\boxed{\mathbf{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t\right]}$$

where c_1, c_2 are constants. Since $\lim_{t\to\infty} e^t = \infty$, the solution approaches infinite as $t \to \infty$.

Problem 5

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x}$$

Find the eigenvalue and eigenvector of this matrix:

$$\begin{aligned} \begin{vmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ 0 & -1 & 1-r \end{vmatrix} &= (1-r)\begin{vmatrix} 1-r & -1 \\ -1 & 1-r \end{vmatrix} - \begin{vmatrix} 2 & -1 \\ 0 & 1-r \end{vmatrix} + \begin{vmatrix} 2 & 1-r \\ 0 & -1 \end{vmatrix} \\ &= (1-r)[(1-r)^2 - 1] - 2(1-r) - 2 \\ &= -(r+1)(r-2)^2 = 0 \end{aligned}$$

$\Rightarrow r_1 = -1, \quad r_2 = r_3 = 2$

Find the eigenvalue when $r_1 = -1$:

$$\begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \boldsymbol{\xi}^{(1)} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$$

Find the eigenvalue when $r_2 = r_3 = 2$:

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Since there is only one eigenvector that corresponds to $r = 2$, we must find a second linearly independent solution.

Let $\mathbf{x} = \boldsymbol{\xi} te^{2t} + \boldsymbol{\eta} e^{2t}$, where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are constant vectors. Substitute this to the original problem:

$$\begin{aligned} \mathbf{x}' &= 2\boldsymbol{\xi} te^{2t} + (2\boldsymbol{\eta} + \boldsymbol{\xi})e^{2t} \\ &= \mathbf{A}(\boldsymbol{\xi} te^{2t} + \boldsymbol{\eta} e^{2t}) \end{aligned}$$

By comparing the coefficients on te^{2t} and e^{2t}, we get the following equations:

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\xi} = \mathbf{0} \qquad (\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$$

From observing these equations, $\boldsymbol{\xi}$ is the eigenvector obtained from the previous part.

$$\begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

By solving this, we get the following solution:

$$\boldsymbol{\eta} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Overall, the general form of the solution will be:

$$\boxed{\mathbf{x} = c_1 \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{2t} + c_3 \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{2t} \right]}$$

Appendix: Python codes to reproduce the plots

Direction field

The following code was used to plot the direction field of §7.5.1(b). The plots from other problems were created by changing the formula inside func ode(t,x) to create the plots. Feel free to change the x and y values to observe what happens for larger/smaller x and y values.

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

def func_ode(t,x):
    dx1dt = 3 * x[0] - 2 * x[1]
    dx2dt = 2 * x[0] - 2 * x[1]
    return np.stack([dx1dt, dx2dt], axis=0)

X, Y = np.meshgrid(np.linspace(-15, 15, 20),
                   np.linspace(-15, 15, 20))

X, Y = X.flatten(), Y.flatten()

dydt = func_ode(None, [X, Y])

fig = plt.figure(figsize=(8,6))
axes=fig.add_subplot(1,1,1)
axes.quiver(X, Y, dydt[0], dydt[1], color='blue')
axes.set_xlim(-10,10)
axes.set_ylim(-10,10)
axes.set_xlabel('x1')
axes.set_ylabel('x2')
axes.set_title('Direction field')
plt.show()
```

Plots of several solutions

The following code was used to create the plot of the solutions. The initial conditions were given as (1,1), (-1,-1) and (2,-2), but they can be changed by changing the elements in x0 list. The function inside func ode(t,x) must be changed to create the plot for the problem. Here, it shows the code for §7.5.1(b). Feel free to change the x, y and t value to observe what happens for larger/smaller x, y and t values.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
%matplotlib inline

def func_ode(t,x):
    dx1dt = 3 * x[0] - 2 * x[1]
    dx2dt = 2 * x[0] - 2 * x[1]
    return np.stack([dx1dt, dx2dt], axis=0)

x0_list = np.array([[1,1], [-1,-1], [2,-2]])
sol_list = []

for x0 in x0_list:
    sol = solve_ivp(func_ode, [0, 2], x0, vectorized = True, dense_output = True)
    sol_list.append(sol)

time = np.arange(0, 2, 0.01)
x_list = []

for sol in sol_list:
    x_list.append(sol.sol(time))

fig, axes = plt.subplots(2, 1, figsize=(12,12))

for x in x_list:
    axes[0].plot(time, x[0])
    axes[1].plot(time, x[1])

axes[0].set_xlabel('t')
axes[0].set_ylabel('x0')
axes[0].set_ylim(-30,30)

axes[1].set_xlabel('t')
axes[1].set_ylabel('x1')
axes[1].set_ylim(-30,30)
```