PS 10 due 4/27 REA RUSTAGI

§7.1: 4, 5, 12, 15
§7.2: 12, 14, 23
§7.3: 6, 25

4) $u^{(4)} - u = 0$ (*)

let: $x_1 = u$ $x_3 = u''$

$x_2 = u'$ $x_4 = u'''$

See that:

$x_1' = x_2$

$x_2' = x_3$

$x_3' = x_4$

$x_4' = u^{(4)}$

Plug into (*): $x_4' - x_1 = 0 \Rightarrow x_4' = x_1$

$$\boxed{\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = x_1 \end{cases}}$$

5) $u'' + 0.25u' + 4u = 2\cos 3t$ $\begin{cases} u(0) = 1 \\ u'(0) = -2 \end{cases}$

Let: $x_1 = u$, $x_2 = u'$

See that: $x_1' = x_2$

$x_2' = u''$

Plug in: $x_2' + 0.25x_2 + 4x_1 = 2\cos 3t$

$$\boxed{\begin{bmatrix} x_1' = x_2 \\ x_2' = 2\cos 3t - 0.25x_2 - 4x_1 \end{bmatrix} \quad \begin{cases} x_1(0) = 1 \\ x_2(0) = -2 \end{cases}}$$

12) (i) $x_1' = -0.5x_1 + 2x_2$ $\begin{cases} x_1(0) = -2 \\ x_2(0) = 2 \end{cases}$
(ii) $x_2' = -2x_1 - 0.5x_2$

a) $x_2 = 0.5x_1' + 0.25x_1$ (iii)
sub into (ii):
$(0.5x_1' + 0.25x_1)' = -2x_1 - 0.5(0.5x_1' + 0.25x_1)$
$= 0.5x_1'' + 0.25x_1' = -2x_1 - 0.25x_1' - 0.125x_1$
$\Rightarrow 0.5x_1'' + 0.5x_1' + 2.125x_1 = 0$
$(\times 8)$: $\boxed{4x_1'' + 4x_1' + 17x_1 = 0}$

b) char eq.: $4r^2 + 4r + 17 = 0$
$r = -0.5 \pm 2i$
$x_1 = e^{-0.5t}[c_1\cos(2t) + c_2\sin(2t)]$

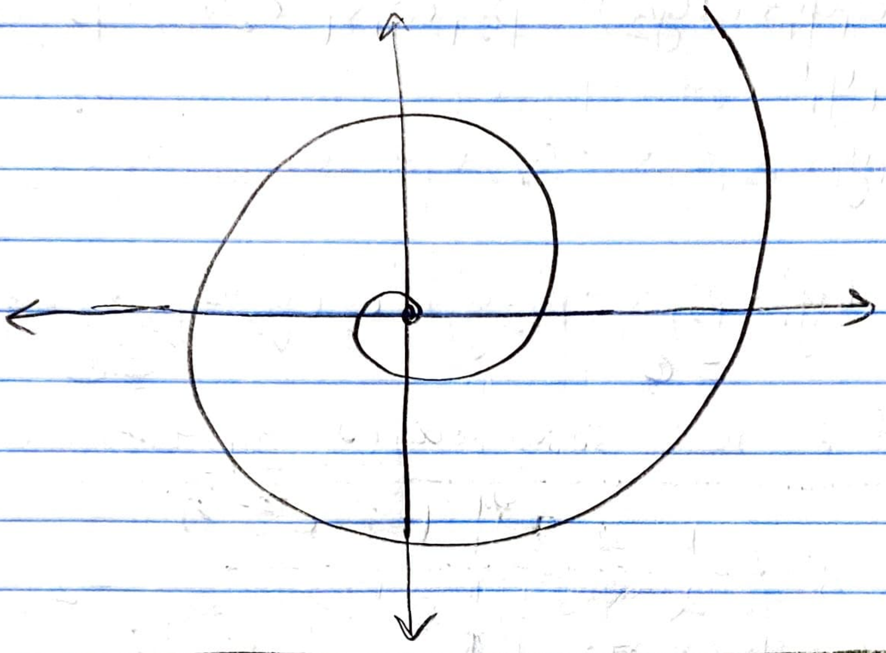
plug into (iii):
$x_2 = 0.5[(\cancel{e^{-0.5t}})[-2c_1\sin(2t) + 2c_2\cos(2t)]$
$+ (c_1\cos(2t) + c_2\sin(2t))(-0.5\cancel{e^{-0.5t}})]$
$+ 0.25(e^{-0.5t}(c_1\cos(2t) + c_2\sin(2t))$

$x_1(0) = c_1 + 0 = -2$, $c_1 = -2$
$x_2(0) = 0.5[2c_2 + (-0.5)c_1] + 0.25(c_1)$
$= c_2 - \cancel{0.25c_1} + \cancel{0.25c_1} = 2$

$\boxed{x_1(t) = e^{-0.5t}[-2\cos(2t) + 2\sin(2t)]}$
$x_2(t) = \frac{1}{2}(e^{-0.5t}(4\sin(2t) + 4\cos(2t)) +$
$(-2\cos(2t) + 2\sin(2t))(-0.5e^{-0.5t}))$
$+ 1/4(e^{-0.5t}(-2\cos(2t) + 2\sin(2t)))$

$$\boxed{x_2(t) = 2e^{-0.5t}(\cos 2t + \sin 2t)}$$

c)



15) $x' = p_{11}(t)x + p_{12}(t)y$ (i)

$y' = p_{21}(t)x + p_{22}(t)y$ (ii)

sub $x = c_1 x_1 + c_2 x_2$ into (i)

$y = c_1 y_1 + c_2 y_2$

$$(c_1 x_1 + c_2 x_2)' = p_{11}(t)(c_1 x_1 + c_2 x_2) + p_{12}(c_1 y_1 + c_2 y_2)$$

$$= c_1 x_1' + c_2 x_2' = p_{11} c_1 x_1 + p_{11} c_2 x_2 + p_{12} c_1 y_1 + p_{12} c_2 y_2$$

$$c_1 x_1' - p_{11} c_1 x_1 - p_{12} c_1 y_1 + c_2 x_2' - p_{11} c_2 x_2 - p_{12} c_2 y_2 = 0$$

$$\Rightarrow c_1\underbrace{(x_1' - p_{11} x_1 - p_{12} y_1)}_{=0} + c_2\underbrace{(x_2' - p_{11} x_2 - p_{12} y_2)}_{=0} = 0$$

b/c $x = x_1, y = y_1$ are sol'ns

and $x = x_2, y = y_2$

sub into (ii):

$$(c_1y_1+c_2y_2)' = p_{21}(c_1x_1+c_2x_2)+p_{22}(c_1y_1+c_2y_2)$$

$$= c_1y_1'+c_2y_2' = p_{21}c_1x_1+p_{21}c_2x_2+p_{22}c_1y_1+p_{22}c_2y_2$$

$$= c_1y_1'-p_{21}c_1x_1-p_{22}c_1y_1+c_2y_2'-p_{21}c_2x_2-p_{22}c_2y_2=0$$

$$\Rightarrow c_1(\underbrace{y_1'-p_{21}x_1-p_{22}y_1}_{=0})+c_2(\underbrace{y_2'-p_2x_2-p_{22}y_2}_{=0})=0$$

for the same reason as before

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12) $$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right)\begin{array}{l} \\ -2(I) \\ -3(I) \end{array}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right)\begin{array}{l} +2(III) \\ \\ \\ \end{array}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -5 & 0 & 2 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right)\begin{array}{l} -3(II) \\ \\ -3(II) \end{array}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 3 & -3 & 1 \end{array}\right)\begin{array}{l} \\ \cdot -1 \\ \cdot -1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right)$$

14) $\begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{pmatrix} = A$

$\det(A) = 1 \cdot \det\begin{pmatrix} 1 & 8 \\ -2 & -7 \end{pmatrix} - 2 \cdot \det\begin{pmatrix} -2 & 8 \\ 1 & -7 \end{pmatrix}$

$+ 1 \cdot \det\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$

$= 1 \cdot (-7+16) - 2 \cdot (14-8) + 1 \cdot (4-1)$

$= 9 - 12 + 3 = \boxed{0} \rightarrow$ singular matrix

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23) $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^t = \begin{pmatrix} e^t + 2te^t \\ 2te^t \end{pmatrix}$

(*) $\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$

$\vec{x}' = \begin{pmatrix} e^t + 2te^t \\ 2te^t \end{pmatrix}' = \begin{pmatrix} e^t + 2te^t + 2e^t \\ 2te^t + 2e^t \end{pmatrix}$

$\begin{pmatrix} 3e^t + 2te^t \\ 2e^t + 2te^t \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} e^t + 2te^t \\ 2te^t \end{pmatrix} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$

$\Downarrow$

$= \begin{pmatrix} 2e^t + 4te^t - 2te^t \\ 3e^t + 6te^t - 4te^t \end{pmatrix} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$

$\checkmark = \begin{pmatrix} 3e^t + 2te^t \\ 2e^t + 2te^t \end{pmatrix}$

$\boxed{\vec{x} \text{ is a sol'n of } (*)}$

6) $x_1 + 2x_2 - x_3 = -2$
$-2x_1 - 4x_2 + 2x_3 = 4$
$2x_1 + 4x_2 - 2x_3 = -4$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ -2 & -4 & 2 & 4 \\ 2 & 4 & -2 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\text{infinite sol'ns where } x_1 = -2 - 2x_2 + x_3}$$

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25) find e-vals, e-vecs of

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

$$(3-\lambda)\cdot \det\begin{pmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix} - 2\cdot \det\begin{pmatrix} 2 & 2 \\ 4 & 3-\lambda \end{pmatrix} + 4\cdot \det\begin{pmatrix} 2 & -\lambda \\ 4 & 2 \end{pmatrix}$$

$$= (3-\lambda)\cdot(-3\lambda + \lambda^2 - 4) - 2\cdot(6 - 2\lambda - 8) + 4\cdot(4 + 4\lambda)$$

$$-9\lambda + 3\lambda^2 - 12 + 3\lambda^2 - \lambda^3 + 4\lambda + 4\lambda + 4 + 16 + 16\lambda$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8$$

$$= (\lambda+1)^2(\lambda-8)$$

$$\boxed{\lambda_{1,2} = -1, \ \lambda_3 = 8}$$ eigenvalues

e-vectors:

$$\lambda = -1 \to \begin{bmatrix} \cancel{4}\,0 & \cancel{2}\,0 & \cancel{4}\,0 \\ 2 & 1 & 2 \\ \cancel{4}\,0 & \cancel{2}\,0 & \cancel{4}\,0 \end{bmatrix}$$

$$\text{e-vecs: } \boxed{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \quad \text{for } \lambda = -1}$$

$$\lambda = 8 \to \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \sim \begin{array}{ccc} \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \\ x2 \quad x1 \quad x2 \end{array}$$

$$\text{e-vec: } \boxed{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \text{for } \lambda = 8}$$