EXISTENCE AND UNIQUENESS FOR SOLUTIONS FOR FIRST ORDER ODES

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Let's begin with an example.

Example 0.1. Solve the ODE

$$y'(t) = y^{2/3}, \qquad y(0) = 0.$$

Well, this should be pretty straightforward since it's separable. We can write it as

$$y^{-2/3}dy = dt.$$

Integrating both sides, we find

$$3y^{1/3} = t + C.$$

Plugging in the initial condition $(0,0)$ shows that C must be zero, so we end up with

$$y = \frac{1}{27}t^3.$$

So we're done, right? But wait a minute. It's always good to think for a minute if there are any obvious or trivial solutions to the ODE. In fact, look: $y(t) \equiv 0$ is another perfectly good solution to the ODE, and it also satisfies the same initial condition $y(0) = 0$. Hmm...

The above example should give you pause. When solving examples in class, we've generally been tacitly assuming that the solutions to first order ODE's are unique once an initial condition is specified. Graphically, this is why the integral curves in the (t,y) plane don't cross each other. Remember, we also used uniqueness to conclude that solutions cannot cross equilibrium values, but rather can only asymptotically approach them. So if first order ODE's *don't* have unique solutions, a lot of our intuition has been wrong...

Luckily, there is some order in the universe, thanks to the following classical theorem by Picard.

Theorem 0.2. *Consider the initial value problem*

$$y'(t) = f(t,y), \qquad y(t_0) = y_0.$$

Assume that $f(t,y)$ *and* $\frac{\partial}{\partial y}f(t,y)$ *are* continuous *functions in a rectangle*

$$R = \{(t,y)\ :\ a < t < b,\ c < y < d\}$$

in the (t,y)*-plane which contains the point* (t_0, y_0)*. Then there* exists *a* unique *solution* $y(t)$ *to the initial value problem, at least for* $t \in [t_0 - h, t_0 + h]$ *with some small* $h > 0$.

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In other words, if $f(t,y)$ and $\frac{\partial}{\partial y}f$ are *continuous*, then we're guaranteed to have a solution and for it to be unique, at least for a short time interval.

Keep in mind that this whole discussion is just for first order ODE's. Also, remember that for *linear* first order ODEs we actually wrote down the *formula* (in terms of some integrals) for the solution. If you go back and check the steps we used to arrive at that formula (using integrating factors), you can see that that formula is the only possible solution, i.e. it's the *unique* solution. So in the *linear case*, we really didn't need any fancy theorem to show us that solutions exist and are unique.

Example 0.3. Let's return to the previous example. For the ODE $y' = y^{2/3}$, we have $f(t,y) = y^{2/3}$. Notice that $f(t,y)$ is actually continuous for all t and y. Namely, it doesn't even depend on t, whereas the graph of $y^{2/3}$ looks like a bent 'V' shape. It has a sharp point at the origin, but it doesn't jump there, so it's continuous. On the other hand, $\frac{\partial}{\partial y}f = \frac{2}{3}y^{-1/3}$ is not even *defined* for $y = 0$, let alone continuous. So Picard's theorem does not apply here, which we already knew since the uniqueness conclusion is violated.

Philosophically, Picard's theorem might feel a little bit unsatisfying, since we like to *solve* equations rather than just have an abstract theorem telling us that they exist. As we've mentioned a few times in class, this is sort of too much to ask for, since equations like $y' = y^2 - t$ simply don't have solutions that can be written in terms of elementary functions or even integrals of elementary functions. However, Picard does have more to offer us! Namely, it turns out that if we iteratively set

$$y_{n+1}(t) := y_0 + \int_{t_0}^{t} f(s, y_n(s))ds,$$

(so $y_1(t) = y_0 + \int_{t_0}^{t} f(s, y_0)$ and so on), then the sequence of functions $\{y_n(t)\}$ actually converges to the solution of the ODE. In order words, as n gets bigger, $y_n(t)$ gets closer and closer to the true solution. This could be useful if we only need an approximate solution. Also, this sequence is the key mathematical idea used to prove Picard's theorem, although the analytical details are beyond the scope of this class.