Midterm 1

Ordinary differential equations
Columbia University Spring 2020
Instructor: Kyler Siegel

Instructions:

- Please write your answers **in this printed exam**. You may use the back of pages for additional work. You may also use printer paper if you need additional space, but you must hand in all relevant work. Please turn in all scratch work which is relevant to your submitted answers.
- Suspected cases of copying or otherwise cheating will be taken very seriously.
- Solve as many problems of the following problems as you can in the allotted time, which is *one hour and fifteen minutes.* I recommend first solving the problems you are most comfortable with before moving on to the more challenging ones. Note that the problems are not ordered by level of difficulty or topic.
- The exams will be graded on a curve. Therefore the raw score is not important, and you do not necessarily need to solve every problem to achieve a good grade. Just do your best!
- Turn off all electronic devices. You may use the restroom if you must, but you may not take any devices with you.
- Good luck!!

Name: ______________________________

Uni: ______________________________

1. (20 points) Find the solution to the initial value problem

$$\begin{cases} y'(t) + sin(t)y(t) = e^t \\ y(3) = 7. \end{cases}$$

You may leave your answer in the form of a definite integral (e.g. something like $\int_{s=2.5}^{s=t} s^{17}\sin(s)ds$).

Solution:

This is a first order linear ODE, so we use the method of integrating factors. Multiplying both sides of the ODE by $\mu(t)$ gives

$$\mu(t)y'(t) + \mu(t)\sin(t)y(t) = \mu(t)e^t.$$

We seek $\mu(t)$ such that the left hand side is $\frac{d}{dt}(\mu(t)y(t))$, i.e. we want

$$\mu(t)y'(t) + \mu(t)\sin(t)y(t) = \mu(t)y'(t) + \mu'(t)y(t),$$

i.e.

$$\frac{\mu'(t)}{\mu(t)} = \sin(t).$$

This is a separable ODE, and integrating gives $\ln|\mu(t)| = -\cos(t) + C$. We just need a single integrating factor, so it suffices to take $\mu(t) = e^{-\cos(t)}$. Then our original ODE becomes

$$\tfrac{d}{dt}(e^{-\cos(t)}y(t)) = e^{-\cos(t)}e^t.$$

We now replace t with a dummy variable s, and integrate both sides from $s = 3$ to $s = t$ to obtain

$$e^{-\cos(t)}y(t) - e^{-\cos(3)}y(3) = \int_{s=3}^{s=t} (e^{-\cos(s)}e^s)ds.$$

We have the initial condition $y(3) = 7$, so this gives

$$y(t) = \frac{7e^{-\cos(3)} + \int_{s=3}^{s=t} e^{s-\cos(s)}ds}{e^{-\cos(t)}} = e^{\cos(t)}\left(7e^{-\cos(3)} + \int_{s=3}^{s=t} e^{s-\cos(s)}ds\right).$$

2. (20 points) Consider the following ODE for a function $y(t)$:

$$y^2e^{ty} + 1 + (e^{ty} + tye^{ty})y' = 0.$$

Find the general solution for $y(t)$. Your answer should involve one arbitrary constant and may be left in implicit form.

Solution:

Note that this ODE is nonlinear and does not appear to be separable, so our best hope is that it's exact. Let's introduce the notation $M(t, y) := y^2e^{ty} + 1$ and $N(t, y) := e^{ty} + tye^{ty}$. We check whether the ODE is closed, i.e. whether $\frac{\partial}{\partial y}M = \frac{\partial}{\partial t}N$. We have

$$\tfrac{\partial}{\partial y}M = 2ye^{ty} + ty^2e^{ty}$$

and

$$\frac{\partial}{\partial t} N = ye^{ty} + ye^{ty} + ty^2 e^{ty},$$

so these do indeed agree. Therefore the ODE is exact, so we seek a function $\psi(t, y)$ such that $\frac{\partial}{\partial t}\psi(t, y) = M(t, y)$ and $\frac{\partial}{\partial u}\psi(t, y) = N(t, y)$. The first equation says

$$\frac{\partial}{\partial t}\psi = y^2 e^{ty} + 1,$$

so by integrating both sides with respect to t we obtain

$$\psi(t, y) = ye^{ty} + t + h(y).$$

Taking the partial derivative of this with respect to y and plugging it into the second equation above, we obtain

$$e^{ty} + tye^{ty} + h'(y) = e^{ty} + tye^{ty},$$

hence it suffices to take $h(y) = 0$. This gives $\psi(t, y) = ye^{ty} + t$, and hence the general solution is given by

$$y(t)e^{ty(t)} + t = 0.$$

Note that there is no reasonable way to solve this as an explicit formula for $y(t)$, so we leave it as an equation describing $y(t)$ implicitly as a function of t.

3. (20 points) Find the solution to the initial value problem

$$\begin{cases} 2y''(t) + 4y'(t) + y(t) = 0 \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$$

Solution:

After making the ansatz $y(t) = e^{rt}$, plugging this into the ODE and simplifying gives the characteristic equation

$$2r^2 + 4r + 1 = 0.$$

The roots are given by $\dfrac{-4 \pm \sqrt{16 - 8}}{4} = -1 \pm \sqrt{2}/2$. Put $r_1 = -1 + \sqrt{2}/2$ and $r_2 = -1 - \sqrt{2}/2$. These are real and distinct roots, so the general solution is given by

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

In order to satisfy the initial conditions, we need

$$\begin{cases} C_1 + C_2 = 0 \\ r_1 C_1 + r_2 C_2 = 1. \end{cases}$$

Solving by substitution, we get

$$r_1 C_1 + r_2(-C_1) = 1,$$

i.e.

$$C_1 = (r_1 - r_2)^{-1} = 1/\sqrt{2},$$

and hence $C_2 = -1/\sqrt{2}$. The solution to the initial value problem is therefore

$$y(t) = \tfrac{1}{\sqrt{2}} e^{(-1+\sqrt{2}/2)t} - \tfrac{1}{\sqrt{2}} e^{(-1-\sqrt{2}/2)t}.$$

4. (I) (10 points) Consider the autonomous ODE $y' = y\sin^2(y)$. Find all equilibrium points and classify them as stable, unstable, or semistable.

Solution: Equilibrium points occur when $y\sin^2(y) = 0$, i.e. when $y = k\pi$ for any integer k. Note that $y\sin^2(y)$ is always nonnegative when y is positive, and always nonpositive when y is negative. From this it follows easily (picture the slope field and integral curve diagrams) that 0 is unstable and $k\pi$ is semistable whenever k is a nonzero integer.

(II) (10 points) Consider the autonomous ODE $y'(t) = (y-1)^3(y-2)^4(y^3-4y^2+3y)$, along with the initial condition $y(0) = y_0$. For each y_0, determine $\lim\limits_{t\to\infty} y(t)$.

Solution: We have $y^3 - 4y^2 + 3y = y(y^2 - 4y + 3) = y(y-1)(y-3)$, so the ODE can be rewritten as

$$y' = y(y-1)^4(y-2)^4(y-3).$$

By drawing the slope field and the corresponding integral curve plot, we find the following limits:

- $\lim_{t\to\infty} y(t) = 0$ if $y_0 < 1$
- $\lim_{t\to\infty} y(t) = 1$ if $1 \leq y_0 < 2$
- $\lim_{t\to\infty} y(t) = 2$ if $2 \leq y_0 < 3$
- $\lim_{t\to\infty} y(t) = 3$ if $y_0 = 3$
- $\lim_{t\to\infty} y(t) = \infty$ if $y_0 > 3$.

For $y_0 > 3$, in fact the solution $y(t)$ has a vertical asymptote, i.e. $\lim_{t\to t_M} y(t) = \infty$ for some $0 < t_M < \infty$ (which depends on y_0). Indeed, for large y, $y(t)$ grows even faster than the solution to the explosion equation $y' = y^2$, which we've seen goes to infinity in finite time. Then the solution $y(t)$ is only defined for $t < t_M$, after which it breaks down, and hence the limit $\lim_{t\to\infty} y(t)$ is not defined for $y_0 > 3$.

5. (20 points) Find the general solution to the ODE $4y'' + 4\lambda y' + (\lambda^2 + 1)y = 0$, where λ is a real-valued parameter. Describe the behavior of $y(t)$ as $t \to \infty$. *Note: your answer should depend on* λ.

Solution:
The characteristic equation is

$$4r^2 + 4\lambda r + (\lambda^2 + 1) = 0,$$

which has roots

$$r = \frac{-4\lambda \pm \sqrt{16\lambda^2 - 16(\lambda^2+1)}}{8} = \frac{-4\lambda \pm 4i}{8} = -\lambda/2 \pm i/2.$$

The general solution is then given by

$$y(t) = C_1 e^{\frac{-\lambda t}{2}} \cos(t/2) + C_2 e^{\frac{-\lambda t}{2}} \sin(t/2).$$

For any λ, there is the trivial solution $y(t) \equiv 0$. Now consider the nontrivial solutions. When $\lambda > 0$, the solutions are oscillatory with amplitude exponentially decaying, and we have $\lim_{t\to\infty} y(t) = 0$ for any C_1 and C_2. When $\lambda = 0$, the general solution is $C_1 \cos(t/2) + C_2 \sin(t/2)$. In this case, the solutions oscillate with period 4π, but with amplitude depending on C_1 and C_2. For $\lambda < 0$, the solutions oscillate with amplitude increasing exponentially. This means that $\lim_{t\to\infty} y(t)$ is not defined, but we have $\lim_{t\to\infty} |y(t)| = \infty$.

6. (0 points) *Bonus problem for up to 4 extra points - do not attempt unless you are confident with all of your other answers!*

Find the general solution to the ODE $y'(t) = y/t + y^2/t^2$ for $t > 0$.

Solution: This example is neither linear, separable, nor exact. However, notice that it has an interesting feature, namely that y/t appears prominently. We make the substitution $v(t) := y(t)/t$. Then we have $y = vt$, and hence $y' = v't + v$. The our ODE becomes the following ODE having $v(t)$ as its dependent variable:

$$v'(t)t + v(t) = v(t) + v(t)^2,$$

i.e.

$$v'(t)t = v(t)^2.$$

This ODE is much simpler and is in now in fact separable:

$$\frac{v'(t)}{v^2(t)} = 1/t,$$

so we get $-1/v(t) = \ln(t) + C$, and hence $v(t) = \dfrac{-1}{\ln(t) + C}$. This gives

$$y(t) = tv(t) = \frac{-t}{\ln(t) + C}.$$

However, notice that this computation assumed that we have $v \neq 0$, i.e. $y \neq 0$. In fact, there is another solution: $y(t) \equiv 0$.