LECTURE 1

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This is a semester long class on ordinary differential equations. Firstly, what is a differential equation? Roughly speaking, it is any equation (or system of equations) involving a function (or collection of functions) and its derivatives. The word *ordinary* means that there is just one independent variable, often denoted by 'x' or 't' and sometimes (but not always) thought of as 'time' in applications. This is in contrast to *partial* differential equations, which involve multiple independent variables and hence partial derivatives with respect to those variables. Examples of ODE's:

(1) $\dfrac{dy}{dx} = x^2 + sin(x)$
(2) $\dfrac{dy}{dx} = x^2 y + x$
(3) $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + \sin(x^7) = 0$
(4) $f'''(x) + f''(x) + 7f'(x) = 0$
(5) $\dot{x} = x^2 t^2$,
(6) $\dfrac{dx}{dt} = x^2 - t$
(7) $\dfrac{dx}{dt} = f(t, x(t)), \qquad x(t_0) = x_0$,

etc. Notice that $\dfrac{dy}{dx}$, $f'(x)$, and $\dot{x}$ are all just different notational conventions for the first derivative, preferred, by Leibniz, Lagrange, and Newton respectively. The equations (1) and (2) are *first order* since they involve only the first derivative, whereas (3) is second order, (4) is third order, etc. Equations (1) and (2) are *linear* whereas (3) is not, since it involves the fancy term $\left(\dfrac{dy}{dx}\right)^2$ (this terminology will come later).

What does it mean to *solve* an ODE? Simply put, it means finding a function, say $y(x)$, which satisfies the specified identity. For example, it is easy to check that the function $y(x) = x^3/3 - \cos(x)$ solves the first equation. In fact, it should be clear that we can find $y(x)$ simply by integrating the right hand side. No such simple prescription exists for the other equations. It seems natural to ask: *does a solution exist? If so, is that solution unique? Perhaps there are two solutions, or three, or infinitely many?*. Note that in example one, we actually have the solution $y(x) = x^3/3 - \cos(x) + C$ for any constant C,

Date: January 21, 2020.

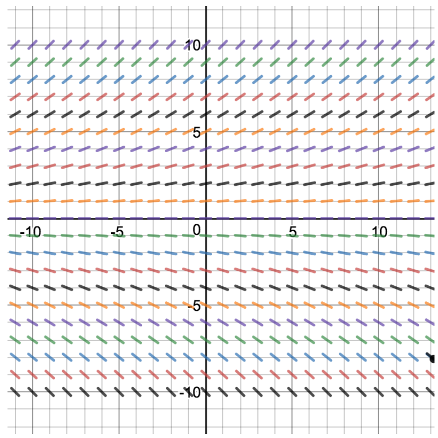

FIGURE 1. Direction field for the ODE $dx/dt = x/10$.

so there are indeed infinitely many solutions. However, if we add the additional constraint that $y(0) = -1$, then we must have $C = 0$ and the solution becomes unique.

ODE's frequently arise as mathematical models of natural phenomena. Let's see some examples.

Example 0.1. (*equation of normal reproduction*) Consider a population of say fish with plenty of food. If $x(t)$ is the number of fish at time t, the equation

$$x'(t) = kx(t)$$

is the 'equation of normal reproduction'. Here $k > 0$ is a constant. This equation says that the rate of reproduction is proportional to the number of fish. We can get more intuition by plotting the *direction field* (a.k.a. *slope field*) in the (t, x) plane, see Figure 1. This shows what slope a solution $x(t)$ would need to have at each point. A curve with the specified slope at each point would constitute the graph of a solution to the given ODE and is sometimes called an *integral curve*. We can see roughly how the solution curves should look by trying to draw them on the grid. We see that if the population starts out as zero, it will be forever zero, i.e. $x(t) \equiv 0$ is a solution to the ODE. If the population starts out positive, it will increase faster and faster. In fact, it could conceivably become infinity after a finite amount of time, but it's hard to tell from the direction field. Our plot also includes negative t and negative x, which make perfect mathematical sense although a negative number of fish doesn't have any interpretation in our model.

To get a more quantitative understanding, we can solve the equation explicitly as follows. Write the equation as

$$\frac{dx}{x} = kdt.$$

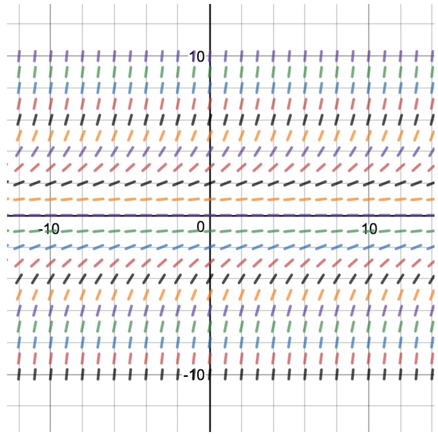

FIGURE 2. Direction field for the ODE $dx/dt = x^2/10$.

Now we integrate both sides, the right side from t_0 to t and the left side from $x(t_0)$ to $x(t)$, obtaining[1]

$$\ln(x(t)) - \ln(x(t_0)) = k(t - t_0).$$

By exponentiating both sides, we obtain

$$x(t) = x(t_0)e^{k(t-t_0)}$$

So indeed, if $x(t_0) = 0$, then $x(t) \equiv 0$ (i.e. no fish will suddenly appear if we don't start with any), whereas for $x(t_0) > 0$, the number of fish at time t is $x(t_0)e^{k(t-t_0)}$ which grows exponentially. Notice that the number of fish never reaches infinity. Also, the number of fish always doubles after an amount of time equal to t_d, where $e^{kt_d} = 2$, i.e. $t_d = \ln(2)/k$. A similar equation (but with $k < 0$) explains why a radioactive isotope has a 'half-life'.

Example 0.2. (*the explosion equation*) Now let's modify the equation to be

$$x' = kx^2,$$

which says that the rate of reproduction is proportional to the number of *pairs* of fish. How does this change things? The direction field, shown in Figure 2, looks qualitatively similar but perhaps grows a bit faster. We can solve this equation using the same trick, rewriting it as

$$\frac{dx}{x^2} = kdt$$

[1]This step might make you slightly nervous, since in calculus courses one is taught that $\frac{dx}{dt}$ is merely a notation and not an actual fraction. In fact, it turns out that this move is perfectly allowed, since dx and dt have rigorous mathematical meaning as *differential forms*. Although these lie beyond the scope of this course, you can just think the above derivation as a heuristic way of discovering a solution. Whenever a heuristic is used to find a supposed solution, you should go back and check that the resulting does indeed satisfy the original ODE, which is usually an easy computation.

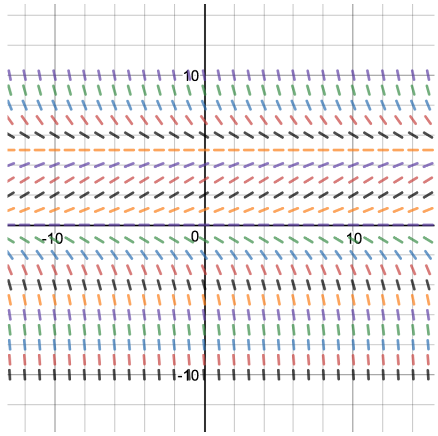

FIGURE 3. Direction field for the ODE $dx/dt = x(5-x)/10$.

and integrating from t_0 to t to obtain

$$-1/x(t) + 1/x(t_0) = k(t - t_0),$$

i.e.

$$x(t) = (1/x(t_0) - k(t - t_0))^{-1}.$$

Let's assume for simplicity that $t_0 = 0$ and put $x_0 = x(0)$, so we get

$$x(t) = \frac{x_0}{1 - ktx_0}.$$

Observe: as t approaches $1/(kx_0)$ from below, $x(t)$ approaches infinity! So the population reaches infinity in finite time, unlike in the previous example. This illustrates that it is dangerous to assume that the solutions to an ODE will exist for all t even if the equation itself exists for all t. Of course it's hard to imagine a lake with infinitely many fish, but there are real world situations that are well-modeled by this equation, at least for some time interval.

Example 0.3. (*reproduction equation in the presence of food competition*) Now let's suppose that our fish have to compete for food, so the effective rate of reproduction diminishes as the number of fish grows. The following ODE attempts to model this:

$$x' = (a - bx)x,$$

where a, b are constants. See Figure 3 for a slope field plot. Now let's set $a = b = 1$ for simplicity, so we get the 'logistic equation':

$$x' = (1 - x)x.$$

The first thing to notice is that $x(t) \equiv 0$ is unsurprisingly a solution, but there is another constant solution (or 'equilibrium solution'), namely $x(t) \equiv 1$. The slope field suggests that if we start with $x(0)$ between 0 and 1, the number of fish will grow asymptotically to one, whereas if we start with $x(0) > 1$, the number of fish will decrease asymptotically to one.

Let's try to verify this explicitly. We write the equation as

$$\frac{dx}{(1-x)x} = dt$$

and integrate, using that an antiderivative for $\frac{1}{(1-x)x}$ is $\ln(x/(1-x))$ (try a partial fraction decomposition) to obtain

$$\frac{x}{1-x} = Ce^t,$$

which can be solved for x as

$$x(t) = \frac{e^t}{C' + e^t}.$$

Note that constant C' is different from C but it makes no difference since they're both just arbitrary constants. The important thing is the relationship between C' and $x(0)$, which we find by plugging in $t = 0$:

$$x(0) = e^0/(C' + e^0),$$

i.e.

$$C' = 1/x(0) - 1.$$

You should convince yourself that for any C' such that $x(0)$ is positive, the solution $x(t)$ is asymptotic to 1 as t approaches infinity, and that it approaches from either below or above depending on whether $x(0)$ is less than 1 or more than 1.

Example 0.4. (*normal reproduction with constant harvesting*) Now consider the ODE

$$x' = ax - b,$$

which represents our fish with normal reproduction rate a, but now we harvest them at a constant rate b (for example we catch fifteen fish per day). We have

$$\frac{dx}{ax - b} = dt,$$

and integrating from 0 to t and putting $x_0 = x(0)$, we have

$$\frac{1}{a}\ln(ax(t) - b) - \frac{1}{a}\ln(ax_0 - b) = t,$$

i.e.

$$\ln\left(\frac{ax - b}{ax_0 - b}\right) = at,$$

i.e.

$$ax - b = (ax_0 - b)e^{at},$$

i.e.

$$x(t) = \frac{b}{a} + (x_0 - b/a)e^{at}.$$

Now let's consider what happens when we plug in rates $a = b = 1$ and initial population $x_0 = 1/2$. Then we get the formula

$$x(t) = 1 - e^t/2.$$

Observe: the population of fish reaches zero after finite time, namely when $t = \ln(2)$. So if we harvest too rapidly, the fish will disappear forever!

We end this lecture with one more conceptual question: *even if an ODE has a solution, can it necessarily be written down in terms of 'familiar' functions?*. Revisiting the examples from the very beginning, in (1) we got lucky and only had to use polynomials and trigonometric functions, but could it be that the solution requires functions that we've never seen before, or even that nobody has ever seen before? Mathematicians have worked hard to prove that various types of ODE's are guaranteed to have solutions. For instance, it is a theorem that the equation (7) always has a unique solution for any given t_0 and x_0, assuming some mild conditions on the function f. On the other hand, Liouville proved that the seemingly innocent equation (6), a special case of (7), has no solution in terms of elementary functions or even their integrals. This might convince you that the general study of ODE's is very complicated, which is true. If you're an aspiring mathematician, you will find this course leads down a rabbit hole with many open problems and active research questions at the other end. At the same time, differential equations show up all over the place in science and engineering. How do people work with them? Firstly, there are many ways to use numerical methods and computer algorithms to get find approximate solutions to ODE's, and sometimes this is sufficient in practical applications. At the same time, there are many special ODE's which can be solved exactly using familiar functions. This will be the focus of this course. In the process we will witness various general phenomena pertaining to ODE's and acquiring an arsenal of intuition and basic techniques which can be used as stepping stones for more complicated ODE's or as basic guiding intuition for numerical methods.