EXAMPLE OF HIGHER ORDER ODE WITH CONSTANT COEFFICIENTS

KYLER SIEGEL

In this note we will explain how to find the general solution to the ODE

$$y^{(7)} - 10y^{(4)} + 25y' = 0.$$

As always, we begin by making the ansatz $y = e^{rt}$ and plugging this into the ODE. After dividing everywhere by e^{rt}, we arrive at the characteristic equation

$$r^7 - 10r^4 + 25r = 0.$$

Firstly, we can easily factor out r to get

$$r(r^6 - 10r^3 + 25) = 0.$$

How can we factor $r^6 - 10r^3 + 25$? According to the rational roots theorem, if there's a (nonzero) rational root it has to be in the set $\{1, 5, 25, -1, -5, -25\}$. However, you can check that none of these give roots. On other hand, notice that

$$r^6 - 10r^3 + 25 = (r^3)^2 - 10(r^3) + 25,$$

and hence it is a perfect square:

$$r^6 - 10r^3 + 25 = (r^3 - 5)^2.$$

This means that the roots of $r^6 - 10r^3 + 25$ are the same as the roots of $r^3 - 5$, except that each one gets repeated twice. Also, recall that by the fundamental theorem of algebra $r^3 - 5$ must have three roots since it's a degree three polynomial, although they might be complex numbers and some of them could potentially be repeated. In fact, as we saw in class, the roots of $r^3 - 5$ are given by $\sqrt[3]{5}$, $\sqrt[3]{5}\exp(2\pi i/3)$, and $\sqrt[3]{5}\exp(4\pi i/3)$. If you are not convinced by this, please verify that each of these numbers raised to the third power gives 5. It will be helpful to notice that

$$(\exp(2\pi i/3))^3 = \exp(2\pi i) = \cos(2\pi) + i\sin(2\pi) = 1.$$

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Therefore, we could write out the roots to $r^7 - 10r^4 + 25r$ as

$$\begin{aligned}
r_1 &= 0 \\
r_2 &= \sqrt[3]{5} \\
r_3 &= \sqrt[3]{5}\exp(2\pi i/3) = \sqrt[3]{5}(\cos(2\pi/3) + i\sin(2\pi/3)) \\
r_4 &= \sqrt[3]{5}\exp(4\pi i/3) = \sqrt[3]{5}(\cos(4\pi/3) + i\sin(4\pi/3)) \\
r_5 &= r_2 \\
r_6 &= r_3 \\
r_7 &= r_4.
\end{aligned}$$

The last three roots are the repeated ones, and notice that the ordering of these roots is completely arbitrary. Also, notice that we can simplify expressions such as $\cos(2\pi/3)$, but it's not so crucial to do so, and it would be impossible if we encountered something more complicated like $\cos(2\pi/7)$.

Finally, let's put together the general solution. We know that it's going to be of the form

$$y(t) = C_1y_1(t) + C_2y_2(t) + C_3y_3(t) + C_4y_4(t) + C_5y_5(t) + C_6y_6(t) + C_7y_7(t),$$

where $y_1(t), ..., y_7(t)$ form a fundamental set of solutions. To get these, we go back to our ansatz e^{rt}, and plug in r_i for r. This gives $y_1(t) = e^{0t} = 1$ and $y_2(t) = e^{\sqrt[3]{5}t}$. Next we get

$$e^{r_3t} = \exp(\sqrt[3]{5}(\cos(2\pi/3) + i\sin(2\pi/3))t).$$

To simplify formulas, let's set $a = \sqrt[3]{5}\cos(2\pi/3)$, $b = \sqrt[3]{5}\sin(2\pi/3)$, $c = \sqrt[3]{5}\cos(4\pi/3)$, and $d = \sqrt[3]{5}\sin(4\pi/3)$.

Since the polynomial $r^3 - 5$ has real coefficients, it must be the case that the non-real roots come in complex conjugate pairs. You should try to convince yourself that this must be true. Also, you should convince yourself that r_3 and r_4 are indeed complex conjugates of each other! In other words, $a = c$ and $b = -d$.

Now we can alternatively write

$$e^{r_3t} = \exp(at + ibt) = \exp(at)\exp(ibt) = \exp(at)(\cos(bt) + i\sin(bt)).$$

This is a perfectly good solution to our original ODE, but it's complex-valued. This is not so bad, but we'd rather find a real-valued fundamental set of solutions, and we know such a thing must exist. If you recall, the real and complex parts of any solution must also be a solution. Therefore, we can take

$$\begin{aligned}
y_3(t) &= \exp(at)\cos(bt) \\
y_4(t) &= \exp(at)\sin(bt).
\end{aligned}$$

Note that we could do the same thing with c, d instead of a, b, but it would give essentially the same solutions we just found.

Finally, to get y_5, y_6, y_7 we do the usual trick of multiplying a repeated root solution by t to get another solution:

$$\begin{aligned} y_5(t) &= t y_2(t) \\ y_6(t) &= t y_3(t) \\ y_7(t) &= t y_4(t). \end{aligned}$$