

## PROBLEM SET #9

**Problem 1.** Suppose we are given  $k \in \mathbb{Z}_{\geq 1}$  and numbers  $n_1, \dots, n_k \in \mathbb{Z}_{\geq 1}$  which are pairwise relatively prime, with product  $N = n_1 \dots n_k$ . For any  $a_1, \dots, a_k$  such that  $0 \leq a_i \leq n_i - 1$  for  $i = 1, \dots, k$ , prove that there is a unique  $x \in \mathbb{Z}_{\geq 1}$  with  $1 \leq x \leq N$  such that the remainder when  $x$  is divided by  $n_i$  is  $a_i$  for  $i = 1, \dots, k$ .

**Problem 2.** (1) How many abelian groups are there of order 1000000 up to isomorphism?  
(2) Write out all abelian groups of order 1000 up to isomorphism in both invariant factor form (see DF §5.2 Theorem 3) and elementary divisor form (see DF §5.2 Theorem 5).

**Problem 3.** Find the number of elements of order 2 and the number of subgroups of index 2 in  $\mathbb{Z}/60 \times \mathbb{Z}/45 \times \mathbb{Z}/12 \times \mathbb{Z}/36$ .

**Problem 4.** Recall that for  $n \in \mathbb{Z}_{\geq 2}$  we have an isomorphism  $\text{Aut}(\mathbb{Z}/n) \cong (\mathbb{Z}/n)^\times$ . If  $n = p^m$  for an odd prime  $p$ , this group is cyclic, whereas if  $n = 2^m$ , this group is isomorphic to  $\mathbb{Z}/2 \times \mathbb{Z}/2^{m-2}$ . For general  $n$ , describe this abelian group in terms of its elementary divisors.

**Problem 5.** Let  $H$  and  $K$  be groups, let  $\phi : K \rightarrow \text{Aut}(H)$  be a homomorphism, and let  $G = H \rtimes_\phi K$  denote the corresponding semidirect product. Prove that we have  $C_G(H) \cap K = \ker \phi$ .

**Problem 6.** Construct a non-abelian group of order 75.

**Problem 7.** Show that there are exactly four possible semidirect products of  $\mathbb{Z}/2$  with  $\mathbb{Z}/8$ . Prove that at least one of these is neither abelian nor isomorphic to  $D_{16}$ .

**Problem 8.** Prove that there are four groups of order 28 up to isomorphism.