PROBLEM SET #8

Problem 1. (1) Let G be a group. By definition, an *automorphism* of G is an isomorphism from G to itself. We denote by $\mathrm{Aut}(G)$ the set of automorphisms of G. Prove that $\mathrm{Aut}(G)$ is a group under composition.

(2) Let p be a prime. Prove that the automorphism group $\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$ is isomorphic to C_{p-1}.

(3) For any prime p and positive integer $n \in \mathbb{Z}_{\geq 1}$, determine the order of the group of automorphisms of $\mathbb{Z}/(p^n\mathbb{Z})$.

Problem 2. (1) Let G be a group of order 231, and let $P \subset G$ be a Sylow 11-subgroup. Prove that P is contained in the center of G.

(2) Let G be a group of order 203, and let H be a normal subgroup of order 7. Prove that H is contained in the center of G.

Problem 3. (1) Let G be a group of order p^2, where p is a prime. Prove that G is isomorphic to either $\mathbb{Z}/(p^2\mathbb{Z})$ or $\mathbb{Z}/(p\mathbb{Z}) \times \mathbb{Z}/(p\mathbb{Z})$. *Note: you should not invoke the classification of finite abelian groups.*

(2) Similarly, let G be an *abelian* group of order p^3, where p is a prime. Prove that G is isomorphic to a direct product of cyclic groups.

Problem 4. Let G be a finite abelian group, and let n be a positive divisor of $|G|$. Prove that G has a subgroup of order n. *Note: you should not invoke the classification of finite abelian groups.*

Problem 5. For a group G, an automorphism is called *inner* if it is given by conjugation by some element $g \in G$. The inner automorphisms of G form a subgroup $\mathrm{Inn}(G) \subset \mathrm{Aut}(G)$. Prove that every automorphism of S_n is inner for $n \geq 2$ and $n \neq 6$. *Hint: follow the outline in Dummit and Foote §4.4 Problem #18.*