MODERN ALGEBRA I
PROBLEM SET 7 SAMPLE SOLUTIONS

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Problem 1. If n_5 and n_7 denote the number of Sylow 5- and 7-subgroups of G respectively, then we have $n_5 \equiv 1 \pmod 5$ and $n_5 | 7 \implies n_5 = 1$, so the unique Sylow 5-subgroup A is normal in G. Similarly $n_7 \equiv 1 \pmod 7$ and $n_7 | 5 \implies n_7 = 1$ and again the unique Sylow 7-subgroup B must be normal in G. Since $\gcd(5, 7) = 1$, it follows that $A \cap B = \{e\}$ and now from Homework 5, Problem 1 we conclude that $G \cong A \times B \cong \mathbb{Z}_5 \times \mathbb{Z}_7 \cong \mathbb{Z}_{35}$.

Problem 2.

1. Any q-cycle generates a subgroup Q of order q, e.g. $(12 \dots q)$;
2. Consider the action of S_q on its Sylow q-subgroups by conjugation. Because q is prime, these all have order q and are generated by any non-identity element. Therefore the number of (Sylow) subgroups of order q is $\frac{q!}{q(q-1)} = (q-2)!$ – there are $q!$ ways to order the elements of a q-cycle, with q cyclic repetitions, and any of its $q-1$ non-trivial powers gives rise to the same subgroup of order q. Since the conjugation action is transitive, this quantity also equals the number of orbits $\implies$ by orbit-stabilizer theorem the normalizer $N_{S_q}(Q)$ of Q has order $\frac{|S_q|}{(q-2)!} = q(q-1)$. Finally $p | q-1 | q(q-1)$, so by Cauchy's theorem $N_{S_q}(Q)$ has a subgroup P of order p;
3. PQ is a group thanks to $P \subseteq N_{S_q}(Q)$, and has order $\frac{|P| \cdot |Q|}{|P \cap Q|} = \frac{pq}{1} = pq$ due to the fact that $\gcd(p, q) = 1$;
4. Pick any $q \in Q$, then $pq = qp \iff pqp^{-1} = q \iff p \in C_{S_q}(q)$. Now the action of S_q on the set of q-cycles by conjugation is transitive, so by orbit-stabilizer theorem $\left|C_{S_q}(q)\right| = \frac{|S_q|}{\#q\text{-cycles}} = \frac{q!}{(q-1)!} = q$. But any $e \neq p \in P$ has order p coprime to q, so we can't have $p \in Q$ as well $\implies p = pe_Q = pe_{S_q}$ and $q = e_P q = e_{S_q} q$ are non-commuting elements of PQ.

Problem 3. See here for Kyler's solution.

Problem 4. We have $1365 = 3 \times 5 \times 7 \times 13$ and if $n_p = 1$ for any of those prime factors p, then the unique Sylow p-subgroup will be a proper normal subgroup, so G automatically cannot be simple. Suppose therefore that $n_p > 1 \ \forall \ p | 1365$; then together with the conditions $n_p \equiv 1 \pmod p$ and $n_p | \frac{1365}{p}$ we deduce that $n_{13} = 105, n_7 \geq 15, n_5 \geq 21$. This already implies that G must have at least $(13-1) \times 105 + (7-1) \times 15 + (5-1) \times 21 = 1434 > 1365$ elements, contradiction.

Problem 5. Note that $\{e\}$ is always its own conjugacy class, so if G were to have only 2 conjugacy classes, then $G \setminus \{e\}$ must constitute a conjugacy class. In such case, the orbit-stabilizer theorem tells us that $|G \setminus \{e\}| \, \big| \, |G|$, i.e. $n - 1 | n$ which is only possible if $n = 2$. Conversely, the group of order 2 is clearly abelian and consequently has precisely 2 conjugacy classes.

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Problem 6. Using the prime factorization $203 = 7 \times 29$, Sylow's theorems imply that $n_{29} \equiv 1 \pmod{29}$ and $n_{29} | 7 \implies n_{29} = 1$, so the unique Sylow 29-subgroup K is normal in G. Since $|H| = \frac{203}{29} = 7$ is coprime to 29, it follows that $H \cap K = \{e\}$ and we conclude from Homework 5, Problem 1 that $G \cong H \times K \cong \mathbb{Z}_7 \times \mathbb{Z}_{29}$.

Problem 7. Note that since $168 = 2^3 \times 3 \times 7$, a Sylow 7-subgroup has order 7 and so do all of its non-identity elements. Conversely, an element of order 7 generates a subgroup of order 7, so it must necessarily lie in a Sylow 7-subgroup. Therefore the number of elements of order 7 is given by the number of non-identity elements in Sylow 7-subgroups, i.e. $6 \times n_7$. We know that $n_7 \equiv 1 \pmod{7}$ and $n_7 | 24$, but also $n_7 \neq 1$ as our group is simple $\implies$ the only possibility left is $n_7 = 8$, in which case the answer to the problem is 48.

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