PROBLEM SET #7

Problem 1. Prove that every group of order 35 is cyclic. You may invoke general results such as Lagrange's theorem and Sylow's theorems but not any specific classification results for groups of given orders.

Problem 2. Suppose that p and q are prime numbers with $p < q$ such that p divides $q-1$. The goal of this problem is to prove that there is a nonabelian group of order pq.

(1) Prove that there is a subgroup Q of S_q of order q.
(2) Prove that the normalizer $N_{S_q}(Q)$ has a subgroup P of order p.
(3) Prove that PQ is a group of order pq.
(4) Prove that PQ is not abelian.

Problem 3. Let G be a finite simple group with $|G| < 100$. Prove that G is either abelian or has order 60.
Hints:

- *Count the elements of G in terms of their orders. When does this exceed $|G$?*
- *If H is a large subgroup of G, consider the action G on the set of left cosets of H in G. What does the first isomorphism theorem say?*
- *Similarly, if H is a Sylow subgroup of G, consider the action G on the set of conjugates of H.*

Problem 4. Prove that a group of order 1365 cannot be simple.

Problem 5. For which $n \in \mathbb{Z}_{\geq 1}$ does there exist a group G of order n with precisely two conjugacy classes?

Problem 6. Suppose that G is a group of order 203 and that H is a normal subgroup of index 29. Prove that G must be abelian.

Problem 7. Let G be a simple group with $|G| = 168$. Determine the number of elements of order 7 in G.