PROBLEM SET #6

Solution 1. Prove that every group of order p^2 for p a prime is abelian. *Hint: what happens when you quotient by the center?*

Problem 1. Let G be a group with $|G| = p^2$. We have seen that the center $Z(G)$ of G must be nontrivial, by an application of the class equation. If $Z(G) = G$, then G is evidently abelian. By Lagrange's theorem, the only other possibility is that we have $|Z(G)| = p$. In this case, note that the quotient group $G/Z(G)$ has order p, and hence is cyclic. Let $xZ(G)$ denote a generator. Then every element of $G/Z(G)$ can be written as $x^k Z(G)$ for some $k \in \mathbb{Z}$, and hence every element of G can be written as $x^k z$ for some $k \in \mathbb{Z}$ and some $z \in Z(G)$.

Let $g = x^k z$ and $g' = x^{k'} z'$ be two such elements in G, with $z, z' \in Z(G)$. To show that G is abelian, it suffices to show that we have $gg' = g'g$. We have

$$gg' = x^k z x^{k'} z' = x^k x^{k'} z z' = x^{k+k'} z' z = x^{k'} x^k z' z = z^{k'} z' x^k z = g' g,$$

as desired.

Solution 2. Let G be a finite group with $|G|$ odd, and let $g \in G$ be an element which is not the identity element. Prove that g and g^{-1} are not conjugate in G. *Hint: suppose by contradiction that g and g^{-1} are conjugate. Consider the conjugacy class of g in G. Show that whenever it contains an element it also contains its inverse.*

Problem 2. Suppose by contradiction that we have $xgx^{-1} = g^{-1}$ for some $x \in G$ and nonidentity $g \in G$. Let C denote the set of all elements in G which are conjugate to g. Note that since $g \neq e$, we have $e \notin C$, since the identity element always lies in a singleton conjugacy class. Suppose that $h \in C$. We claim that $h^{-1} \in C$ as well. Indeed, $h \in C$ means that we have $h = aga^{-1}$ for some $a \in G$. Then $h^{-1} = ag^{-1}a^{-1} = axgx^{-1}a^{-1} = (ax)g(ax)^{-1}$, which shows that h^{-1} is also conjugate to g, and hence lies in C.

Since the elements of C appear in pairs (h, h^{-1}) with $h \neq h^{-1}$, it follows that $|C|$ must be even. By the orbit stabilizer theorem, $|C| = |G|/|C_G(g)|$, where $C_G(g)$ denotes the centralizer of g in G. But this implies that $|G|$ be must be even, whereas by hypothesis $|G|$ is odd.