PROBLEM SET #6

Problem 1.

(1) Give two permutations $\sigma, \sigma' \in A_n$ for some $n \geq 1$ such that σ and σ' have the same cycle type, but σ and σ' are *not* conjugate in A_n.

(2) Prove that if n is odd, the set of n-cycles in A_n consists of two conjugacy classes of equal size.

Problem 2. Prove that every group of order p^2 for p a prime is abelian. *Hint: what happens when you quotient by the center?*

Problem 3. Determine the decomposition of D_8 into conjugacy classes. You should give a representative of each conjugacy class and state how many elements are in that conjugacy class. Check that the class equation holds.

Problem 4. Determine the decomposition of S_5 into conjugacy classes. You should give a representative of each conjugacy class and state how many elements are in that conjugacy class. Check that the class equation holds.

Problem 5. Let $G \times X \to X$ be an action of a finite group G on a set X, and let r denote the number of orbits of this action. For each $g \in G$, let $X_g := \{x \in X \ : \ gx = x\}$ denote the set of elements in X fixed by g.

(1) Prove that we have

$$\sum_{g \in G} |X_g| = \sum_{x \in X} |\mathrm{stab}(x)|.$$

(2) Prove that we have

$$\sum_{x \in X} \frac{1}{|\mathrm{orb}(x)|} = r.$$

(3) Prove that we have

$$r = \frac{1}{|G|} \sum_{g \in G} |X_g|.$$

This is known as Burnside's Lemma.

Problem 6. Suppose we are given an equilateral triangle T. We wish to count the number N of distinguishable ways of coloring each of the three sides of T with one of four colors (say blue, red, green, and yellow), where:

- We are allowed to use the same color more than once (e.g. we can paint two sides with yellow and the third side with blue).
- Two such colorings are considered indistinguishable if we can transform one to another by a rotation in three space (i.e. an element of the dihedral group D_6). For example, coloring the three sides by red, blue, and green respectively is the same thing as coloring them by blue, red, and green respectively, since they differ by flipping the triangle over.

If we remember the ordering of the three sides, there are $4^3 = 64$ different possible colorings. Let X denote the corresponding set of colorings (i.e. $|X| = 64$), and consider the natural action of S_3 on X. Convince yourself that N is precisely the number of orbits of this action.

(1) For each $\sigma \in S^3$, compute $|X_\sigma|$ as defined in the previous problem.
(2) Use Burnside's lemma to show that $N = 20$.

Problem 7. Let G be a finite group with $|G|$ odd, and let $g \in G$ be an element which is not the identity element. Prove that g and g^{-1} are not conjugate in G. *Hint: suppose by contradiction that g and g^{-1} are conjugate. Consider the conjugacy class of g in G. Show that whenever it contains an element it also contains its inverse.*