PROBLEM SET #5 SOLUTIONS

Problem 1. Let G be a group and let A and B be normal subgroups such that $AB = G$ and $|A \cap B| = 1$. Prove that G is isomorphic to $A \times B$.

Solution 1. We define a set map $\Phi : A \times B \to G$ by $(a, b) \mapsto a \cdot b$. We claim that this is a homomorphism. To verify this, we need

$$\Phi((a, b) \cdot (a', b')) = \Phi((a, b)) \cdot \Phi((a', b')).$$

We have $\Phi((a, b) \cdot (a', b')) = \Phi((aa', bb')) = aa'bb'$ and $\Phi((a, b)) \cdot \Phi((a', b')) = aba'b'$, so it suffices to show that we have $aa'bb' = aba'b'$ for any $a, a' \in A$ and $b, b' \in B$. By left multiplying this equality by a^{-1} and right multiplying by $(b')^{-1}$, this is equivalent to showing that $a'b = ba'$, or equivalently $a'b(a')^{-1}b^{-1} = e$, where e denotes the identity element of G. Since B is a normal subgroup of G, we have $a'b(a')^{-1} \in a'B(a')^{-1} = B$, and hence $a'b(a')^{-1}b^{-1} \in Bb^{-1} = B$. Similarly, since A is a normal subgroup of G, we have $b(a')^{-1}b^{-1} \in bAb^{-1} = A$, and hence $a'b(a')^{-1}b^{-1} \in a'A = A$. Therefore we have $a'b(a')^{-1}b^{-1} \in A \cap B = \{e\}$, so $a'b(a')^{-1}b^{-1} = e$, as desired.

By our formula for the cardinality of a subset of the form AB, we have $|G| = |A||B|/|A \cap B| = |A||B| = |A \times B|$. Therefore G and $A \times B$ have the same cardinarity. Moreover, Φ is surjective, since any element g of G can be written as $g = ab$ for some $a \in A$ and $b \in B$, and hence $\Phi((a, b)) = ab = g$. It follows that Φ is also injective, and hence a bijectiion.

Problem 2. Let G be a group, and let A and B be normal subgroups of G such that $G = AB$. Prove that $G/(A \cap B)$ is isomorphic to $(G/A) \times (G/B)$.

Solution 2. By the second isomorphism theorem, $A \cap B$ is a normal subgroup of B and we have $AB/A \cong B/(A \cap B)$. Similarly, $A \cap B$ is a normal subgroup of A, and we have $AB/B \cong A/(A \cap B)$. In general, if G, G', H, H' are groups such that G is isomorphic to G' and H is isomorphic to H', then $G \times H$ is isomorphic to $H' \times G$. Therefore $(G/A) \times (G/B)$ is isomorphic to $A/(A \cap B) \times B/(A \cap B)$.

Observe that in the quotient group $\overline{G} := G/(A \cap B)$, $\overline{A} := A/(A \cap B)$ and $\overline{B} := B/(A \cap B)$ are subgroups such that $\overline{G} = \overline{AB}$. Indeed, each element of $\overline{G}$ can be written as $abA \cap B$ for some $a \in A$ and $b \in B$, and we have $abA \cap B = (aA \cap B)(bA \cap B) = \overline{A}\,\overline{B}$. Also, an element $gA \cap B \in \overline{G}$ for $g \in G$ lies in $\overline{A} \cap \overline{B}$ if and only if $g \in A \cap B$, which means that $|\overline{A} \cap \overline{B}| = |(A \cap B)/(A \cap B)| = 1$.

It is also the case that $\overline{A}$ is a normal subgroup of $\overline{G}$, since for any cosets $\overline{g} = gA \cap B \in \overline{G}$ and $\overline{a} = aA \cap B \in \overline{A}$ represented by $g \in G$ and $a \in A$, we have $\overline{g}\,\overline{a}\,\overline{g}^{-1} = gag^{-1}A \cap B$, and this lies in $\overline{A}$ since $gag^{-1} \in A$ by normality of A in G. Similarly, $\overline{B}$ is a normal subgroup of $\overline{G}$.

We can now apply the previous problem to the group $\overline{G}$ and its subgroups $\overline{A}, \overline{B}$ to conclude that $\overline{G}$ is isomorphic to $\overline{A} \times \overline{B}$, i.e. $G/(A \cap B)$ is isomorphic to $A/(A \cap B) \times B/(A \cap B) \cong (G/A) \times (G/B)$.

Problem 3. Let G be a group and let A and B be subgroups such that we have $A \trianglelefteq B$ and $B \trianglelefteq G$. Do we have $A \trianglelefteq G$?

Solution 3. Note that $A \trianglelefteq B$ means that $bAb^{-1} = A$ for any $b \in B$, and $B \trianglelefteq G$ means that $gBg^{-1} = B$ for any $g \in G$, whereas $A \trianglelefteq G$ means that $gAg^{-1} = A$ for any $g \in G$. There is no obvious reason why A should be normal in G. Since all subgroups of abelian groups are normal, in order to find a counterexample we need to think about nonabelian groups.

Consider the dihedral group D_8, where r denotes a counterclockwise rotation by 90 degrees and s denotes a reflection about the line of symmetry through the first vertex. Consider the subgroups $A := \langle s \rangle = \{e, s\}$ and $B := \langle r^2, s \rangle = \{e, r^2, s, sr^2\}$. Since A has index two in B, it is a normal subgroup of B. Similarly, since B has index two in G, it is a normal subgroup of G. However, A is not a normal subgroup of G. Indeed, recall that we have the relation $srs = r^{-1}$, and hence $rsr^{-1} = sr^{-2}$. Note that sr^{-2} is not equal to e or s, since it sends the first vertex to the third one, whereas e and s both fix the first vertex. Then $s \in A$ but $sr^{-2} \notin A$, so A cannot be a normal subgroup of G.