PROBLEM SET #5

Problem 1. Let G be a group and let A and B be normal subgroups such that $AB = G$ and $|A \cap B| = 1$. Prove that G is isomorphic to $A \times B$.

Problem 2. Let G be a group, and let A and B be normal subgroups of G such that $G = AB$. Prove that $G/(A \cap B)$ is isomorphic to $(G/A) \times (G/B)$.

Problem 3. For $n \in \mathbb{Z}_{\geq 1}$ and $\sigma \in S_n$, the *cycle type* of σ is obtained by writing the cycle decomposition of σ and then writing the lengths $n_1, \ldots, n_r$ of the resulting cycles in nondecreasing order $n_1 \leq \cdots \leq n_r$. Here r is the number of cycles in the cycle decomposition of σ. Note that we have $n_1 + \cdots + n_r = n$, and we say that $n_1, \ldots, n_r$ is a *partition* of n. For example, if $n = 7$ and $\sigma = (1\ 3\ 2)(4\ 6)(5)(7)$, then the cycle type of σ is $1, 1, 2, 3$.

Prove that two elements of S_n are conjugate if and only if they have the same cycle type. Conclude that the number of conjugacy classes of S_n equals the number of partitions of n.

Problem 4. Let G be a group and let A and B be subgroups such that we have $A \trianglelefteq B$ and $B \trianglelefteq G$. Do we have $A \trianglelefteq G$?

Problem 5. List all possible composition series for Q_8 and D_8. How many are there for each? What are the composition factors in each case? You do not need to rigorously justify your answer.

Problem 6. Describe a normal subgroup N of S_4 of index 6. What is the isomorphism type of the quotient group S_4/N? Describe the lattice of subgroups of S_4 containing N, and match these with the lattice of subgroups of S_4/N as in the Fourth Isomorphism Theorem.

Problem 7. Prove that every finite group G is isomorphic to a subgroup of the general linear group $\mathrm{GL}_n(\mathbb{C})$ for some $n \in \mathbb{Z}_{\geq 1}$.

Problem 8. Prove that every nontrivial finite group has a composition series.

Problem 9. Prove the following special case of the Jordan–Hölder theorem: if G has a composition series $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_r = G$ and another one of the form $1 = M_0 \trianglelefteq M_1 \trianglelefteq M_2 = G$, then $r = 2$. Moreover, the (unordered) list of composition factors is the same in both cases. *Hint: use the second isomorphism theorem.*

Problem 10 (Optional - not to be graded)**.** Exercise 9 in §3.2 of Dummit–Foote walks you through a nice proof of Cauchy's Theorem.