Modern Algebra I Fall 2019
Problem Set 4 Sample Solutions

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Problem 1.

(a). Let s and r be the reflection and rotation by angle $\frac{\pi}{2}$ respectively, then

$$D_8 = \langle s, r \mid s^2 = r^4 = 1, rs = sr^{-1} \rangle = \{r^i \mid 0 \le i \le 3\} \sqcup \{sr^i \mid 0 \le i \le 3\}.$$

Using these relations, it follows that for each element of the form sr^i we have for example $(sr^i)r = sr^{i+1}$ and $r(sr^i) = sr^{i-1} \neq (sr^i)r$, so $sr^i \notin Z(D_8)$. Analogously, if $r^i \in Z(D_8)$, then $r^i s = sr^i \iff sr^{-i} = sr^i \iff i = 0$ or 2. Conversely, it is easy to see that $1, r^2 \in Z(D_8)$ and therefore $Z(D_8) = \{1, r^2\}$.

(b). We deduce from above that the quotient group $D_8/Z(D_8)$ has order $8 \div 2 = 4$, and since it has no element of order 4, it must be isomorphic to $C_2 \times C_2$.

Problem 2.

(a).

$$\sigma_1 := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$

decomposes into cycles as $(16)(2345)$, which expands as $(16)(25)(24)(23)$. This is a product of an even number of transpositions, so $\sigma_1 \in A_6$.

(b).

$$\sigma_2 := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 2 & 1 & 5 \end{pmatrix}$$

decomposes into cycles as (142365), which expands as $(15)(16)(13)(12)(14)$. This is a product of an odd number of transpositions, so $\sigma_2 \notin A_6$.

(c).

$$\sigma_3 := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$$

decomposes into cycles as $(14)(25)(36)$. This is a product of an odd number of transpositions, so $\sigma_3 \notin A_6$.

Problem 3.

(a). To show that $C_G(A)$ is a subgroup of G, we must show that $C_G(A)$ contains the identity, is closed under multiplication, and contains inverses.

- Clearly $1 \in C_G(A)$, because for any $a \in A$ we have $1a1^{-1} = a$.
- Suppose $g_1,\, g_2 \in C_G(A)$; let $a \in A$ be any element.
  We know that $g_1 a g_1^{-1} = a$ and $g_2 a g_2^{-1} = a$, so

$$\begin{aligned}(g_1 g_2)\,a\,(g_1 g_2)^{-1} &= (g_1 g_2)\,a\,(g_2^{-1} g_1^{-1}) \\ &= g_1 (g_2 a g_2^{-1})\, g_1^{-1} \\ &= g_1 a g_1^{-1} \\ &= a\end{aligned}$$

  hence $g_1 g_2 \in C_G(A)$.
- Suppose $g \in C_G(A)$; and let $a \in A$ be any element.
  We know that $gag^{-1} = a$, so

$$g^{-1}ag = g^{-1}(gag^{-1})g = (g^{-1}g)\,a\,(gg^{-1}) = a$$

  hence $g^{-1} \in C_G(A)$.

Thus, $C_G(A)$ is a subgroup of G. //

However, $C_G(A)$ is not necessarily a normal subgroup of G; for instance, if $G = S_3$ and $A = \{(12)\}$, then $C_G(A) = \{\text{id}, (12)\}$, which we know is not normal in S_3.

[Remark: We have the following:

Prop: If $N_G(A) = G$, then $C_G(A) \trianglelefteq G$.

Proof: To show that $C_G(A)$ is normal in G, we must show for any $g \in G$ and $h \in C_G(A)$ that $ghg^{-1} \in C_G(A)$. Let $a \in A$ be any element; we know that $hah^{-1} = a$, and from the hypothesis $N_G(A) = G$, we have $b := g^{-1}ag \in A$ for any $g \in G$. Thus:

$$\begin{aligned}(ghg^{-1})\,a\,(ghg^{-1})^{-1} &= (ghg^{-1})\,a\,(gh^{-1}g^{-1}) \\ &= gh(g^{-1}ag)h^{-1}g^{-1} \\ &= ghbh^{-1}g^{-1} \\ &= gbg^{-1} \\ &= a\end{aligned}$$

hence $ghg^{-1} \in C_G(A)$, and $C_G(A)$ is a normal subgroup of G. //

For instance, since G is always normal in itself, we have $N_G(G) = G$, so this implies that the center of G, defined by $\mathcal{Z}(G) := C_G(G)$, is always a normal subgroup of G.]

(b). A straightforward computation shows:

$$\begin{aligned}
C_{S_3}(\{\text{id}\}) &= S_3 \\
C_{S_3}(\{(12)\}) &= \{\text{id}, (12)\} \\
C_{S_3}(\{(13)\}) &= \{\text{id}, (13)\} \\
C_{S_3}(\{(23)\}) &= \{\text{id}, (23)\} \\
C_{S_3}(\{(123)\}) &= \{\text{id}, (123), (132)\} \\
C_{S_3}(\{(132)\}) &= \{\text{id}, (123), (132)\}
\end{aligned}$$

For instance, to compute $C_{S_3}(\{(123)\})$, we observe that (123) obviously commutes with itself, so $(123) \in C_{S_3}(\{(123)\})$; as $C_{S_3}(\{(123)\})$ is a group, this means $\langle(123)\rangle \subseteq C_{S_3}(\{(123)\})$. Now we simply check

$$\begin{aligned}
(123)(12) &= (13) \neq (23) = (12)(123) \\
(123)(13) &= (23) \neq (12) = (13)(123) \\
(123)(23) &= (12) \neq (13) = (23)(123)
\end{aligned}$$

so $(12), (13), (23) \notin C_{S_3}(\{(123)\})$; and $C_{S_3}(\{(123)\}) = \langle(123)\rangle$, as claimed. [In fact, since in general $C_G(A) = \bigcap_{a \in A} C_G(\{a\})$, these determine $C_{S_3}(A)$ for any $A \subset S_3$.]

Problem 4.

(a). Let g and h be arbitrary elements of $N_G(A)$, then

$$(gh^{-1})a(gh^{-1})^{-1} = g(h^{-1}ah)g^{-1} \in gAg^{-1} = A \;\forall a \in A,$$

so $gh^{-1} \in N_G(A)$ and the normalizer of A is indeed a subgroup. It need not be a normal subgroup as will be seen below.

(b). If σ is a 2-cycle, then $N_{S_3}(\langle\sigma\rangle) = \langle\sigma\rangle$ which is not normal, thus proving (a). Otherwise $N_{S_3}(\sigma) = S_3$ because the subgroup generated by σ will be normal.

Problem 5.

(a). From the multiplication rule, we see that the center is $\mathcal{Z}(Q_8) = \{1, -1\}$.

(b). We have the following:

<u>Claim</u>: The subgroups of Q_8 are $\langle 1\rangle$, $\langle -1\rangle$, $\langle i\rangle$, $\langle j\rangle$, $\langle k\rangle$, and Q_8.

Proof: First observe that $\langle -i\rangle = \langle i\rangle$ and $\langle -j\rangle = \langle j\rangle$ and $\langle -k\rangle = \langle k\rangle$, so we have covered every subgroup of Q_8 generated by one element.

Now let $A \subset Q_8$ be any subset with at least two elements, and consider the subgroup H generated by A. Either H can in fact be generated by one element, or we can take two 'non-redundant' elements $a, b \in A$ (*viz.* $b \notin \langle a\rangle$ and $a \notin \langle b\rangle$) with $\langle a, b\rangle \subseteq H$.

But we have $ij = k$, so $k \in \langle i, j\rangle$, and $\langle i, j\rangle = \langle i, j, k\rangle = Q_8$. Thus, $\langle \pm i, \pm j\rangle = Q_8$. Similarly, we check that $\langle \pm i, \pm k\rangle = Q_8$ and $\langle \pm j, \pm k\rangle = Q_8$, so in every non-redundant case, we have checked that $\langle a, b\rangle = Q_8$.

Of course, if H is a subgroup of Q_8 with $\langle a, b\rangle = Q_8 \subseteq H$, then it must be the case that $H = Q_8$.

This finishes the proof that each of the proper subgroups of Q_8 can be generated by a single element; hence the list of subgroups of Q_8 is as claimed. //

Now we prove that every subgroup of Q_8 is normal in Q_8.

- In any group G, the trivial subgroups $\{\text{id}\}$ and G itself are normal. Thus, $\{1\}$ and Q_8 are normal in Q_8.
- In any group G, the center $\mathcal{Z}(G)$ is normal (cf. the remark in part (a) of problem 3). Thus, $\mathcal{Z}(Q_8) = \{1, -1\}$ is normal in Q_8.
- In any group G, any subgroup H with $|G| = 2|H|$ is normal, because the left cosets in G/H are H and $G-H$, and the right cosets in $H\backslash G$ are also H and $G-H$; therefore, if $g \in H$ then $gH = H = Hg$, and if $g \notin H$ then $gH = G - H = Hg$, so left and right cosets are the same for all $g \in G$.

  Thus, $\langle i\rangle$, $\langle j\rangle$, $\langle k\rangle$, each of order 4, are normal in Q_8.

Hence every subgroup of Q_8 is normal in Q_8.

(c). Clearly Q_8 is not abelian, so Q_8 cannot be isomorphic to any of the abelian groups C_8, $C_2 \times C_4$, $C_2 \times C_2 \times C_2$.

Also, Q_8 has only one element with order 2 (namely, -1), so Q_8 cannot be isomorphic to $D_{2\cdot 4} := \langle r, s \mid r^4 = s^2 = 1, rs = sr^{-1}\rangle$, which has five such elements (namely, r^2, s, sr, sr^2, sr^3).

Hence Q_8 is not isomorphic to any of these other subgroups of order 8.

Problem 6.

For $\Phi : G \to H$, let $K \coloneqq \ker(\Phi)$ and $J \coloneqq \mathrm{im}(\Phi)$. We claim the following:

Claim: $K \trianglelefteq G$.

Proof: To show that K is a normal subgroup of G, we must show for any $g \in G$ and $k \in K$ that $gkg^{-1} \in K$. We know that Φ is a homomorphism and that $\Phi(k) = 1$, so we have:

$$\Phi(gkg^{-1}) = \Phi(g)\Phi(k)\Phi(g)^{-1} = \Phi(g)\Phi(g)^{-1} = 1$$

so $gkg^{-1} \in K$. Hence K is normal in G. //

Thus, G/K is a group; we can now define a map $\varphi : G/K \to J$ by $\varphi(gK) = \Phi(g)$. We check that φ is well-defined: if $g_1, g_2 \in G$ are such that $g_1K = g_2K$, then there exists $k \in K$ with $g_1k = g_2$, so

$$\varphi(g_1K) = \Phi(g_1) = \Phi(g_1)\Phi(k) = \Phi(g_1k) = \Phi(g_2) = \varphi(g_2K)$$

hence φ is well-defined.

Now we check that φ is an isomorphism between G/K and J.

- Suppose $g_1K, g_2K \in G/K$ are any two elements. Then

  $$\varphi(g_1K)\varphi(g_2K) = \Phi(g_1)\Phi(g_2) = \Phi(g_1g_2) = \varphi((g_1g_2)K)$$

  so φ is a homomorphism.

- Suppose $g_1K, g_2K \in G/K$ are such that $\varphi(g_1K) = \varphi(g_2K)$. Then

  $$\Phi(g_1g_2^{-1}) = \Phi(g_1)\Phi(g_2)^{-1} = \varphi(g_1K)\varphi(g_2K)^{-1} = 1$$

  so $g_1g_2^{-1} \in K$, which means $g_1K = g_2K$. Hence φ is an injection.

- Suppose $j \in J$ is any element. Then there exists $g \in G$ such that $\Phi(g) = j$. Thus, $\varphi(gK) = \Phi(g) = j$, so $j \in \mathrm{im}(\varphi)$. Hence φ is a surjection.

Thus, we have shown that $\varphi : G/K \to J$ is an isomorphism. Hence $G/K \cong J$, as desired. ■

Problem 7. If $\Phi : S_3 \to G$ is a homomorphism, then $\ker\Phi$ is a normal subgroup of S_3 and $|\,\mathrm{im}\,\Phi| = \frac{|S_3|}{|\ker\Phi|}$. Since the only normal subgroups of S_3 are $\{1\}, \{(123)\}, S_3$, it follows that the only possible values for $|\,\mathrm{im}\,\Phi|$ are $6, 2, 1$. Conversely, each normal subgroup N can be realized as the kernel of the canonical map $\Phi : S_3 \to S_3/N$, hence all of the values above can be achieved.