PROBLEM SET #4

Problem 1. (a) Determine the center $Z(D_8)$ of the dihedral group D_8.[1]
(b) Give the multiplication table for the quotient group $D_8/Z(D_8)$. Which familiar group is this isomorphic to?

Problem 2. For each of the following permutations in S_6, write out its cycle decomposition, and determine whether or not it is an element of the alternating subgroup A_6.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 4 & 5 & 2 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 6 & 2 & 1 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$

Problem 3. (a) Given a group G and a subset $A \subset G$, the *centralizer* of A in G is defined by $C_G(A) := \{g \in G \ : \ gag^{-1} = a \text{ for all } a \in A\}$. Prove that $C_G(A)$ is a subgroup. Is it a normal subgroup? You should rigorously justify your answer.[2]
(b) For each element $x \in S_3$, determine the centralizer of $\{x\}$.

Problem 4. (a) Given a group G and a subset $A \subset G$, the *normalizer* of A in G is defined by $N_G(A) := \{g \in G \ : \ gAg^{-1} = A \text{ for all } a \in A\}$. Prove that $N_G(A)$ is a subgroup. Is it a normal subgroup? You should rigorously justify your answer.
(b) For each element $x \in S_3$, determine the normalizer of $\langle x \rangle$.

Problem 5. The *quaternion group* Q_8 is a group of order eight, defined as a set by $\{1, -1, i, -i, j, -j, k, -k\}$. The group multiplication is defined such that we have

$$(-1) \cdot (-1) = 1, \quad i \cdot j = k, \quad j \cdot i = -k, \quad j \cdot k = i.$$
$$k \cdot i = j, \quad i \cdot k = -j, \quad 1 \cdot k = k, \quad k \cdot (-1) = -k,$$

and so on. You might be able to guess the full pattern from this, but see §1.5 in Dummit and Foote for the full definition of the multiplication structure. You may take for granted that this defines a group, but you should convince yourself.

(a) Determine the center of Q_8.

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[1] Recall that $D_8 = D_{2 \cdot 4}$ is the group of symmetries of the square. You may represent the elements using the notation from lectures, using matrix representations, or using any other notation, but you should carefully explain your notation.

[2] *Note to students*: as a rule of thumb, on the problem sets you should always provide rigorous justification for your answers unless the instructions explicitly say otherwise. I will sometimes add phrases like "you should rigorously justify your answer" just for emphasis. You should try to prove everything from scratch as much as possible, to a reasonable extent. You may also cite results from lectures or the textbooks that was already covered, provided that you already understand the proofs they do not directly trivialize the material at hand. When in doubt you should use your own discretion, or ask the instructor / TAs for clarification.

(b) List all normal subgroups of Q_8. You do not need to rigorously justify your answer.

(c) Prove that Q_8 is not isomorphic to C_8, $C_2 \times C_4$, $C_2 \times C_2 \times C_2$, or D_8.

Problem 6. Give a proof of the *First Isomorphism Theorem* (sometimes called the *Fundamental Theorem of Homomorphisms.*:

> If G and H are groups and $\Phi : G \to H$ is a homomorphism, then $\ker(\Phi)$ is a normal subgroup of G, and the quotient group $G/\ker(\Phi)$ is isomorphic to $\operatorname{im}(\Phi)$.

Problem 7. Let $\Phi : S_3 \to G$ be a homomorphism from the symmetric group S_3 to some group G. What are the possible values for $|\operatorname{im}(\Phi)|$, the order of the image? You should rigorously justify your answer.