PROBLEM SET #3

Problem 1. Describe all of the homomorphisms from $\mathbb{Z}/(12\mathbb{Z})$ to C_{12}.[1] How many are there? How many of them are isomorphisms?

Problem 2. Let G be a group and let $x \in G$ be an element. Prove that for any $k \in \mathbb{Z}_{\geq 1}$, we have $|x^k| = \frac{|x|}{\gcd(|x|,k)}$. *Recall that $|x| = \mathrm{ord}(x)$ denotes the order of x as an element of G.*

Problem 3. Recall that we defined the dihedral group $D_{2\cdot 5}$ (a.k.a. D_{10}) to be the subgroup of $O(2)$ consisting of those orthogonal matrices which preserve the regular pentagon. We will label the vertices in counterclockwise order as v_1, v_2, v_3, v_4, v_5, and we assume that $v_1 = (1, 0)$.

(1) For each element of $D_{2\cdot 5}$, write its representation as a permutation in S_5. You may use for example the notation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ as a shorthand the permutation $v_1 \mapsto v_2$, $v_2 \mapsto v_3$, $v_3 \mapsto v_4$, $v_4 \mapsto v_5$, $v_5 \mapsto v_1$.
(2) Determine the order of each element of $D_{2\cdot 5}$.
(3) Determine all numbers which appear as the index of some subgroup of $D_{2\cdot 5}$. *Note: you should justify your answer. Keep in mind that for this type of problem Lagrange's theorem gives only a necessary condition.*
(4) Is $D_{2\cdot 5}$ abelian?
(5) Pick a subgroup of order two, and write out the corresponding partition of $D_{2\cdot 5}$ into cosets.

Problem 4. Suppose that G and H are finite groups of the same order which are isomorphic.

(1) Assume that G is a cyclic group. Prove that H is also a cyclic group.
(2) Assume that G is an abelian group. Prove that H is also an abelian group.

Note: for this problem please work directly with the definitions of cyclic group, abelian group, and isomorphism, without invoking any results we stated during class.

Problem 5. Show directly that the group $(\mathbb{Z}/(20\mathbb{Z}))^\times$ is isomorphic to a direct product of cyclic groups, i.e. it is isomorphic to $\mathbb{Z}/(k_1\mathbb{Z}) \times \cdots \times \mathbb{Z}/(k_n\mathbb{Z})$ for some $n \in \mathbb{Z}_{\geq 1}$ and $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 2}$.

Problem 6. We define SO(3) to be the group of 3×3 orthogonal[2] matrices whose determinant is 1. This is the group of rotations in three-space, and you can visualize each element as a rotation about some axis by some angle.

[1] Recall that for $n \in \mathbb{Z}_{\geq 1}$, C_n is the group of complex nth roots of unity, with binary operation given by ordinary multiplication of complex numbers.

[2] Recall that an orthogonal matrix is a square matrix A such that $A \cdot A^T$ and $A^T \cdot A$ are both the identity matrix.

(1) Check that SO(3) satisfies the three axioms of a group. *You may take for granted that matrix multiplication is associative, as well as any standard properties of transposes and determinants.*
(2) Prove that any reflection about a two-plane (or rather the matrix representation of such a linear transformation) is *not* included in SO(3). *Hint: what is the determinant of such a matrix? Note that after a change of basis you can take the plane to be the xy-plane.*
(3) Show that SO(3) is nonabelian.
(4) Consider the cube in $\mathbb{R}^3$ whose set of eight vertices is $\{(i,j,k)\ :\ i,j,k \in \{1,-1\}\}$. Let $H \subset \mathrm{SO}(3)$ be the subgroup consisting of those rotations which map this cube to itself setwise.[3] What is the order of H? *You should provide some justification for your answer.*
(5) Observe that each element of H determines a permutation of the set of 8 vertices. Give an example of such a permutation which does *not* arise from an element of H. *Note: if you wish to identify such a permutation with an element of S_8, you will first need to choose an ordering of the vertices.*
(6) Similarly, each element of H determines a permutation of the set of 6 faces. Give an example of such a permutation which does *not* arise from an element of H.

[3]Here *setwise* means that each point in the cube gets mapped to another point in the cube, but the individual points of the cube might get shuffled around.