PROBLEM SET #1

Problem 1. How many partitions are there of the set $\{1, 2, 3, 4, 5, 6\}$?

Solution 1. The number of partitions of a set with n elements ($n = 6$ in this case) is called a *Bell number*. These numbers show up all the time in different parts of mathematics (see for example https://en.wikipedia.org/wiki/Bell_number as a starting point). To count, let's first count the number of different types of parit,itions. For example, if a partition has one part of size 3 and three parts of size 1, we'll write $3 + 1 + 1 + 1$. The possibilities are:

(1) 6
(2) 5+1
(3) 4+2
(4) 3+3
(5) 4+1+1
(6) 3+2+1
(7) 2+2+2
(8) 3+1+1+1
(9) 2+2+1+1
(10) 2+1+1+1+1
(11) 1+1+1+1+1+1.

Now, for example, how many partitions of type $2 + 2 + 1 + 1$ are there? We have: $\{1, 2\} \sqcup \{3, 4\} \sqcup \{5\} \sqcup \{6\}$ and $\{1, 3\} \sqcup \{2, 5\} \sqcup \{4\} \sqcup \{6\}$ and so on.[1] Notice that for each permutation $\sigma \in S_6$ we can consider the partition

$$\{\sigma(1), \sigma(2)\} \sqcup \{\sigma(3), \sigma(4)\} \sqcup \{\sigma(5)\} \sqcup \{\sigma(6)\}.$$

In other words, starting with our first type $2+2+1+1$ partition $\{1, 2\} \sqcup \{3, 4\} \sqcup \{5\} \sqcup \{6\}$, we get lots of other partitions of type $2+2+1+1$ by permuting the elements. In fact, it's easy to see that *every* type $2 + 2 + 1 + 1$ permutation comes in this form for some choice of $\sigma \in S_6$. This shows that there are at most $|S_6| = 6!$ partitions of type $2 + 2 + 1 + 1$.

However, there will be lots of redundancies. For one thing, if we permute the individual entries of any given part, we get the same partition. For example,

$$\{1, 2\} \sqcup \{3, 4\} \sqcup \{5\} \sqcup \{6\} = \{2, 1\} \sqcup \{4, 3\} \sqcup \{5\} \sqcup \{6\}.$$

For each part of size k there are $|S_k| = k!$ different ways of ordering the elements of that part. This means that for each part of size k, there is a $k!$-fold redundancy.

Moreover, if we reorder the parts of a given size, we get the same partition. For example,

$$\{1, 2\} \sqcup \{3, 4\} \sqcup \{5\} \sqcup \{6\} = \{3, 4\} \sqcup \{1, 2\} \sqcup \{6\} \sqcup \{5\}.$$

[1] Here $\sqcup$ means "disjoint union", i.e. it means the same thing as $\cup$ but it's used to emphasize that we're taking a union of disjoint sets (and of course the parts of a partition are required to be disjoint).

If there are l parts of size k, there are $|S_l| = l!$ different ways of ordering these parts, so we get an $l!$-fold redundancy.

Overall, this means there are

$$\frac{6!}{(2!2!1!1!)(2!2!)} = 45$$

partitions of size $2+2+1+1$. Using this method, we can add up the partitions of each of the 11 different types to get 203 in total. Later, we'll see that the language of group theory, and in particular *group actions*, is very helpful for this type of counting problem.