Midterm 1

Modern Algebra 1
Columbia University Fall 2019
Instructor: Kyler Siegel

Instructions:

- Please write your answers **in this printed exam**. You may use the back of pages for additional work. You may also use printer paper if you need additional space, but you must hand in all relevant work. Please turn in all scratch work which is relevant to your submitted answers.
- Suspected cases of copying or otherwise cheating will be taken very seriously.
- Solve as many problems of the following problems as you can in the allotted time, which is *one hour and fifteen minutes.* I recommend first solving the problems you are most comfortable with before moving on to the more challenging ones. Note that the problems are not ordered by level of difficulty or topic.
- The exams will be graded on a curve. Therefore the raw score is not important, and you do not necessarily need to solve every problem to achieve a good grade. Just do your best!
- There are twenty true or false questions, four multiple choice questions, four short answer questions, and two short proof questions.
- For true or false questions, you will receive $+2$ points for a correct answer, 0 points for no answer, and -3 points for an incorrect answer. For multiple choice questions, you will receive $+4$ points for a correct answer, 0 points for no answer, and -2 points for an incorrect answer. This means **you should not make random guesses** unless you are reasonably sure that you know the answer.
- You may use any commonly used notation for standard groups, subgroups, and their elements, as long as it is completely unambiguous. If you are using nonstandard notation you must fully explain it for credit.
- Turn off all electronic devices. You may use the restroom if you must, but you may not take any devices with you.
- Good luck!!

Name: ____

Uni: ____

Question:	1	2	3	4	Total
Points:	30	16	18	20	84
Score:					

Notation reminders:

- $N_G(A) := \{g \in G \ : \ gAg^{-1} = A\}$ denotes the normalizer of a subset $A \subset G$.
- $\ker(\Phi)$ and $\mathrm{im}\,(\Phi)$ denote the kernel and image respectively of a homomorphism Φ
- $D_{2\cdot n}$ denotes the dihedral group corresponding to the symmetries of the regular n-gon.

1. *True or false questions. Circle one. You do not need to provide any justification.*

(I) (2 points) Every group of order 19 is abelian.
A. True B. False

Solution: True. Since 19 is a prime, every group of order 19 is in fact cyclic.

(II) (2 points) The group $\mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(3\mathbb{Z})$ is isomorphic to $\mathbb{Z}/(6\mathbb{Z})$
A. True B. False

Solution: True. One can check directly that $([1],[1])$ has order 6, so the group is cyclic. Later this will also follow from the Chinese remainder theorem.

(III) (2 points) The group $\mathbb{Z}/(3\mathbb{Z}) \times \mathbb{Z}/(3\mathbb{Z})$ is isomorphic to $\mathbb{Z}/(9\mathbb{Z})$
A. True B. False

Solution: False. It is easy to show that every nonidentity element in the former has order at 3, in contrast to $\mathbb{Z}/(9\mathbb{Z})$.

(IV) (2 points) If G is a group and H and K are subgroups, then $HK = \{hk \ : \ h \in H, \ k \in K\}$ is a subgroup of G.

A. True B. False

Solution: False. We saw a counterexample in class: take $G = S_3$, $H = \langle (1\ 2) \rangle$, and $K = \langle (1\ 3) \rangle$.

(V) (2 points) Every permutation $\sigma \in S_{100}$ can be written as a product of 3-cycles.

A. True B. False

Solution: False. Every product of 3-cycles lies in the alternating group, which is a proper subgroup.

(VI) (2 points) $(\mathbb{Z}/11\mathbb{Z})^\times$ is a cyclic group.

Solution: True. We've already stated (but not yet proved) that $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic whenever p is prime. Or we could directly look for a generator.

A. True B. False

(VII) (2 points) A group all of whose proper subgroups are cyclic is cyclic.

A. True B. False

Solution: False. The Klein four group gives a simple counterexample.

(VIII) (2 points) If G and H are groups and $\Phi : G \to H$ is a homomorphism, then we have $N_G(\ker(\Phi)) = G$.

A. True B. False

Solution: True. This is equivalent to the kernel being a normal subgroup, which is always the case.

(IX) (2 points) Suppose that H is a subgroup of a group G, and that H is an abelian group. Then H is a normal subgroup of G.

A. True B. False

Solution: False. A simple counterexample is given by $G = S_3$ and $H = \langle(1\ 2)\rangle$.

(X) (2 points) If H is a subgroup of a finite group G with $|G| = 2|H|$, then H is a normal subgroup of G.

A. True B. False

Solution: True. We saw in class that index two subgroups are automatically normal.

(XI) (2 points) If G is a group of order four, then it is isomorphic to either $(\mathbb{Z}/(5\mathbb{Z}))^\times$ or $(\mathbb{Z}/(8\mathbb{Z}))^\times$.

A. True B. False

Solution: True. Notice that the first group is cyclic, while the second is not (we saw this in class, or you can directly check that no element has order four). We've seen many times that every group of order four is either cyclic or isomorphic to the Klein four group.

(XII) (2 points) If H and K are subgroups of a group G, and we have $H \subset N_G(K)$, then $HK = KH$.

A. True B. False

Solution: True. This was stated as a lemma in class (and is a corollary in Dummit and Foote).

(XIII) (2 points) There exists a surjective homomorphism from $D_{2\cdot 5}$ to $\mathbb{Z}/(2\mathbb{Z})$.

A. True B. False

Solution: True. Since $D_{2\cdot 5}$ has a subgroup of order five, which is necessarily normal, the corresponding projection homomorphism does the trick, since the quotient group has order two and hence is isomorphic to $\mathbb{Z}/(2\mathbb{Z})$.

(XIV) (2 points) The dihedral group $D_{2\cdot 3}$ is isomorphic to the symmetric group S_3.
A. True B. False

Solution: True. We can always view $D_{2\cdot n}$ as a subgroup of S_n, and in the case $n = 3$ they both have order six so they must coincide.

(XV) (2 points) If H and K are normal subgroups of G, then $H \cap K$ is also a normal subgroup of G.
A. True B. False

Solution: True. Basically, for $g \in G$, $g(H \cap K)g^{-1}$ lies in H because H is normal, and in K because K is normal, so it lies in $H \cap K$.

2. *Multiple choice questions. You do not need to provide any justification. In each case,* ***select all that apply****.*

(I) (4 points) Which of the following groups is isomorphic to $(\mathbb{Z}/(10\mathbb{Z}))^\times$?
A. $\mathbb{Z}/(10\mathbb{Z})$ B. $\mathbb{Z}/(4\mathbb{Z})$ C. $C_2 \times C_2$ D. $(\mathbb{Z}/(8\mathbb{Z}))^\times$

Solution: B. The elements are $[1], [3], [7], [9]$. We can check that for example $[3]$ generates the group, so its cyclic. Note that $(\mathbb{Z}/(8\mathbb{Z}))^\times$ is not cyclic.

(II) (4 points) What is the remainder when 13^{101} is divided by 17?
A. 1 B. 3 C. 11 D. 13

Solution: D. We have $[13]^2 = [169] = [-1]$, so $[13]^{101} = [-1]^{50} \cdot [13] = [13]$.

(III) (4 points) Which of the following groups is isomorphic to a subgroup of $D_{2\cdot 8}$?
A. C_2 B. $C_2 \times C_2$ C. C_8 D. $C_2 \times C_8$

Solution: A,B,C. The subgroups isomorphic to C_2 and $C_2 \times C_2$ can be found using reflections, and the basic rotation generates a cyclic subgroup of order 8. D_{16} is not abelian, so this rules out $C_2 \times C_8$.

(IV) (4 points) Which of the following groups is isomorphic to the center of $D_{2\cdot 5}$?

A. C_1 B. C_2 C. $C_2 \times C_2$ D. C_4

Solution: A. For dihedral groups $D_{2\cdot n}$ with n *odd*, the center is trivial. Using the representation $D_{2\cdot 5} = \langle r, s \,:\, r^5 = 1, s^2 = 1, rs = sr^{-1}\rangle$, the elements can be written as $1, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4$. One can then check easily that only 1 commutes with both r and s.

3. *Short answer questions. You do not need to provide any justification for full credit. However, if you do you might receive some partial credit if your answer is incorrect but well-reasoned.*

(I) (6 points) How homomorphisms are there from C_8 to C_{14}?

Solution: Two. Let a and b denote generators of C_8 and C_{14} respectively. Such a homomorphism Φ is uniquely determined by $\Phi(a)$. There are 14 possibilities for $\Phi(a)$. However, many of these will not define valid homomorphisms. In order for this to be well-defined, it's necessary to have $\Phi(a^8) = 1$, which means we must have $\Phi(a)^8 = 1$, and one can show this is also sufficient. The elements of C_{14} are $1, b, b^2, \ldots, b^{13}$, and the order of b^k is $14/\gcd(k, 14)$. We need $14/\gcd(k, 14)$ to be a divisor of 8, and this occurs when $k = 0, 7$. That is, there's one homomorphism with $\Phi(a) = 1$, and another one with $\Phi(a) = b^7$.

(II) (6 points) Let $\sigma \in S_6$ be the permutation given by $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 1 & 6 & 3 & 2 \end{pmatrix}$. Write σ as a product of transpositions, and also determine the sign of σ.

Solution: By following along the permutation, we find $1 \mapsto 4 \mapsto 6 \mapsto 2 \mapsto 5 \mapsto 3 \mapsto 1$, so this gives a single cycle $\sigma = (1\ 4\ 6\ 2\ 5\ 3)$. This can be written as $(1\ 3)(1\ 5)(1\ 2)(1\ 6)(1\ 4)$. Since there are 5 transpositions, the sign is -1, i.e. it is an odd permutation.

(III) (6 points) Explain at least one way in which the following statement of the Second Isomorphism Theorem is mathematically incorrect:

> Let G be a group, and let A and B be subgroups. Then AB is a subgroup of G, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ and $AB/B \cong A/A \cap B$.

Solution: We are missing the assumption $A \subset N_G(B)$. Without this, AB is not necessarily even a subgroup of G. For example, if $G = S_3$, $A = \langle(1\ 2)\rangle$, and $B = \langle(1\ 3)\rangle$, we saw in class that AB is not a subgroup of S_3.

4. *Short proofs. Make your arguments as rigorous as possible. You may cite results covered in class provided you are completely clear about what you are citing.*

(I) (10 points) Let G and H be finite groups such that $\gcd(|G|, |H|) = 1$. Let $\phi : G \to H$ be a homomorphism. Prove that ϕ sends every element of G to the identity element of H.

Solution: By the first isomorphism theorem, we have $G/\ker(\phi) \cong \mathrm{im}\,(\phi)$. In particular, we have $|G/\ker(\phi)| = |G|/|\ker(\phi)| = |\mathrm{im}\,(\phi)|$. This shows that $|\mathrm{im}\,(\phi)|$ is a divisor of $|G|$. On the other hand, since $\mathrm{im}\,(\phi)$ is a subgroup of H, by Lagrange's theorem we have that $|\mathrm{im}\,(\phi)|$ is a divisor of $|H|$. Then $|\mathrm{im}\,(\phi)|$ divides both $|G|$ and $|H|$ and hence it must divide their greatest common divisor, which is 1. This shows that $|\mathrm{im}\,(\phi)| = 1$, i.e. $\mathrm{im}\,(\phi) = \{e\}$, where e is the identity element of H. This means that ϕ sends every element of G to the identity element of H

(II) (10 points) Let G be a finite cyclic group of order n, and let k be an integer which is relatively prime to n. Prove that for any element $g \in G$, there exists an element $h \in G$ such that $g = h^k$. *For 3 bonus points: prove that the same holds without assuming that G is cyclic.*

Solution: By a corollary of the Euclidean algorithm we can find integers x and y such that $xn + yk = \gcd(n, k) = 1$. Also, for any $g \in G$, we have $g^n = e$, where e is the identity element in G. Indeed, this is a standard corollary of Lagrange's theorem: we have that $|g| = |\langle g \rangle|$ divides n, and hence $g^n = g^{(|g|)(n/|g|)} = e^{n/|g|} = e$. We therefore have

$$g = g^1 = g^{xn+yk} = (g^n)^x (g^y)^k = e^x (g^y)^k = (g^y)^k.$$

Then putting $h := g^y$, we have $g = h^k$ as desired.

In fact, this proof did not assume G to be cyclic, so it works for any finite group.