Midterm 1
Modern Algebra 1
Columbia University Fall 2019
Instructor: Kyler Siegel

Exam instructions (these will also be printed on the actual exam):

- Please write your answers **in this printed exam**. You may use the back of pages for additional work. You may also use printer paper if you need additional space, but you must hand in all relevant work. Please turn in all scratch work which is relevant to your submitted answers.
- Suspected cases of copying or otherwise cheating will be taken very seriously.
- Solve as many problems of the following problems as you can in the allotted time, which is *one hour and fifteen minutes*. I recommend first solving the problems you are most comfortable with before moving on to the more challenging ones. Note that the problems are not ordered by level of difficulty or topic.
- The exams will be graded on a curve. Therefore the raw score is not important, and you do not necessarily need to solve every problem to achieve a good grade. Just do your best!
- There are twenty true or false questions, four multiple choice questions, four short answer questions, and two short proof questions.
- For true or false questions, you will receive $+2$ points for a correct answer, 0 points for no answer, and -3 points for an incorrect answer. For multiple choice questions, you will receive $+4$ points for a correct answer, 0 points for no answer, and -2 points for an incorrect answer. This means **you should not make random guesses** unless you are reasonably sure that you know the answer.
- You may use any commonly used notation for standard groups, subgroups, and their elements, as long as it is completely unambiguous. If you are using nonstandard notation you must fully explain it for credit.
- Turn off all electronic devices. You may use the restroom if you must, but you may not take any devices with you.

Notation reminders:

- $N_G(A) := \{g \in G \ : \ gAg^{-1} = A\}$ denotes the normalizer of a subset $A \subset G$.
- $\ker(\Phi)$ and $\mathrm{im}\,(\Phi)$ denote the kernel and image respectively of a homomorphism Φ
- $D_{2\cdot n}$ denotes the dihedral group corresponding to the symmetries of the regular n-gon.

Some sample questions:

1. *True or false questions. Circle one. You do not need to provide any justification.*

(I) Every subgroup of a cyclic group is normal.
A. True B. False

Solution: True. Since cyclic groups are abelian, every subgroup is left invariant by conjugation.

(II) There exists a homomorphism from the symmetric group S_4 to the symmetric group S_5 such that the kernel consists only of the identity element.

A. True B. False

Solution: True. This is equivalent to having an injective homomorphism from S_4 to S_5. We can simply send each permutation in S_4 to the one in S_5 which acts on $1, 2, 3, 4$ in the same way and leaves 5 fixed.

(III) The symmetric group S_4 admits a surjective homomorphism to $\mathbb{Z}/(5\mathbb{Z})$

A. True B. False

Solution: False. By the first isomorphism theorem, this would imply that 5 divides $|S_4| = 4!$, which it doesn't.

(IV) If G and H are groups and $\Phi : G \to H$ is a homomorphism, then we have $N_H(\mathrm{im}\,(\Phi)) = \mathrm{im}\,(\Phi)$.

A. True B. False

Solution: False. This would mean that there are no elements of H which normalize $\mathrm{im}\,(\Phi)$ except for elements of $\mathrm{im}\,(\Phi)$ itself. But this is not necessarily true. For example, $\mathrm{im}\,(\Phi)$ could be a proper normal subgroup of H, in which case we have $N_H(\mathrm{im}\,(\Phi)) = H \neq \mathrm{im}\,(\Phi)$.

(V) If G is a group of order eight, then it is isomorphic to C_8, $C_2 \times C_4$, $C_2 \times C_2 \times C_2$, or $D_{2\cdot 4}$.

A. True B. False

Solution: False. The quaternion group Q_8 is not isomorphic to anything in this list.

(VI) The subgroup of S_5 generated by $(1\ 3\ 2)(4)(5)$ is normal.
A. True B. False

Solution: False. One can check directly, but the point is that we can conjugate $(1\ 3\ 2)$ to any other 3-cycle.

(VII) Let G be a group with at least two distinct elements, whose only subgroups are G itself and the trivial subgroup $\{e\}$. Then $|G|$ is a prime number.
A. True B. False

Solution: True. If we pick any nonidentity element, it generates a cyclic subgroup, which is apparently the whole group. If G were not prime, we could find proper nontrivial subgroups by basic properties of cyclic groups.

2. *Short answer questions. You do not need to provide any justification for full credit. However, if you do you might receive some partial credit if your answer is incorrect but well-reasoned.*

(I) List all of the subgroups of S_3, and indicate which ones are normal.

Solution: Let $e \in S_3$ denote the identity permutation. The subgroups of S_3 are $\langle(1\ 2)\rangle$, $\langle(1\ 3)\rangle$, $\langle(2\ 3)\rangle$, $\langle(1\ 2\ 3)\rangle$, and $\{e\}$. Only the last two are normal subgroups.

3. *Short proofs. Make your arguments as rigorous as possible. You may cite results covered in class provided you are completely clear about what you are citing.*

(I) Prove that every subgroup of a finite cyclic group is cyclic.

Solution:
Let G be a finite cyclic group, and H an arbitrary subgroup of G. Our goal is to show that H is a cyclic group. Let x be a generator of G. Note that every element in H is of the form x^k for some $k \in \mathbb{Z}_{\geq 0}$ (we only need to consider nonnegative powers since G is finite). Let $k \in \mathbb{Z}_{\geq 0}$ be minimal such that $x^k \in H$.

Put $y := x^k$. We claim that y generates H. Indeed, suppose otherwise by contradiction. Then we can find some $l \in \mathbb{Z}_{>0}$ such that $x^l \in H$ and x^l is not a power of y, which means that l is not divisible by k. Since l is not divisible by k, by the division algorithm we can write $l = ak + b$ for some $a, b \in \mathbb{Z}_{\geq 0}$ with $1 \leq b < k$. But then since $x^k \in H$ and $x^{ak+b} \in H$, we also have

$$(x^k)^{-a}x^{ak+b} = x^b \in H,$$

using the fact that H is closed under multiplication and taking inverses. This contradicts the minimality property in the definition of k.