FINAL EXAM TOPICS

The final exam will cover all of the material from this semester. There will be a slight emphasis on the newer material covered in the last few weeks, and a slight de-emphasis of the very early material (e.g. definition of a group, which I hope you have mastered by now), but overall the questions will be designed to evaluate your understanding of the key group theory concepts and techniques covered throughout the course.

Regarding the format, you should expect to see something similar to the previous exams, but longer. That is, a mixture of various types of problems: true / false, multiple choice, short answers, and short proofs.

For a reminder of the topics covered before the first and second midterms, you can consult the topics lists posted on the course website. In this document I'll focus on the main topics covered since the second midterm. These are:

- **direct products**: the definition, recognizing direct products, internal versus external direct products
- **the classification of finitely generated abelian groups**: the statement, invariant factor and elementary divisor normal forms, converting from one form to another, counting abelian groups of a given order, a bit on the proof (to the extent we covered in class)
- **automorphisms**: the definition, determining automorphism groups of various groups, relationship to conjugation actions
- **semidirect products**: the definition, recognizing semidirect products, examples of semidirect products, listing all possible semidirect products for small groups
- **classification results**: classifying all groups of a given small order, classifying groups with simple prime factorizations

Here are a few sample problems. These are not intended to be representative of the final exam questions. Rather, I am including them merely to give an extra source of practice. These are of mixed difficulty. For more sample problems (apart from of course the previous problem sets and midterms) I recommend working through various exercises in the relevant sections of Dummit and Foote and also Fraleigh.

Problem 1. *How many abelian groups are there of order* 7200*, up to isomorphism?*

Problem 2. *Are the groups* $\mathbb{Z}/(20\mathbb{Z}) \times \mathbb{Z}/(10\mathbb{Z}) \times \mathbb{Z}/(5\mathbb{Z}) \times \mathbb{Z}/(7\mathbb{Z})$ *and* $\mathbb{Z}/(14\mathbb{Z}) \times \mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(2\mathbb{Z}) \times \mathbb{Z}/(5\mathbb{Z}) \times \mathbb{Z}/(5\mathbb{Z}) \times \mathbb{Z}/(5\mathbb{Z})$ *isomorphic?*

Problem 3. *Is the quaternion group* Q_8 *isomorphic to a semidirect product of two groups, neither of which is trivial (i.e. has order one)? How about* S_4*? How about* $(\mathbb{Z}/(5\mathbb{Z}))^\times$*?*

Problem 4. *Prove that every finite abelian group is isomorphic to a direct product of cyclic groups.*

Problem 5. *How many groups are there of order* 12 *up to isomorphism? Describe each possibility.*

Problem 6. *Prove that* D_{12} *is isomorphic to* $S_3 \times \mathbb{Z}/(2\mathbb{Z})$.

Problem 7. *Let* p *be a prime. What is the order of the automorphism group of* $\mathbb{Z}/(p\mathbb{Z}) \times \mathbb{Z}/(p\mathbb{Z})$*?*

Problem 8. *Give both invariant factor and elementary divisor normal forms of the abelian group* $(\mathbb{Z}/(1000\mathbb{Z}))^{\times}$.

Problem 9. *How many groups are there of order* 125*, up to isomorphism?*

Problem 10. *How many semidirect products (not necessarily up to isomorphism) are there of the form* $\mathbb{Z}/(15\mathbb{Z}) \rtimes \mathbb{Z}/(2\mathbb{Z})$. *Prove that every group of order* 30 *is isomorphic to one of these.*

Problem 11. *Prove that there are exactly five groups of order* 20 *up to isomorphism.*

Problem 12. *Classify groups of order* 253 *up to isomorphism.*