Introduction to Mathematics: Week 2 Handout

September 17, 2019

1 Proof by Contradiction

To do a proof by contradiction, you assume the statement you want to prove is false, and then show that that assumption leads to a logical inconsistency, which is therefore a contradiction, and so the original statement is true.

The contradiction will be an intermediate statement that we arrive to using sound logical steps within the proof, but which we know is false (e.g. deriving the statement $0 = 1$).

In other words: if we want to show P is true, we assume $\sim P$ is true and find a contradiction. Therefore, since P must either be true or false we get that P is true.

Exercise 1. Prove that there is no greatest integer.

Exercise 2. Show that $\sqrt{2}$ is irrational.

Exercise 3. Prove that there are no integers x and y such that $x^2 = 4y + 2$.

Definition 1. A natural number p is prime if $p > 1$ and p is divisible only by 1 and p.

Exercise 4. The set of prime numbers is infinite.

Definition 2. Say a function $f : X \to Y$ is *injective* if for all $a, b \in X$, $f(a) = f(b)$ only when $a = b$.

Exercise 5. Suppose $f : X \to Y$ is a function such that $f(f(x)) = x$ for all $x \in X$. Prove that f is injective.

2 Proof by Contrapositive

If we want to show $P \Rightarrow Q$, it is equivalent to show $(\sim Q) \Rightarrow (\sim P)$.
A proof of the contrapositive goes as following:

1. Suppose $x \in D$ such that $Q(x)$ is false.
2. Show that $P(x)$ is false
3. Then by the contrapositive, P(x) is true $\Rightarrow$ Q(x) is true.

Exercise 6. Show using the contrapositive that if the square of an integer is even, then the integer is even.

Exercise 7. Prove that for all integers a and b, if $a + b$ is odd then a is odd or b is odd.

Exercise 8. Prove that for every prime number p, if $p \neq 2$ then p is odd.

Definition 3. a is divisible by b if $\exists c$ such that $a = bc$

Definition 4. Say two integers a, b are *relatively prime* if there are no primes p such that a and b are divisible by p.

Exercise 9. Prove that if x^2, y^2 are relatively prime, then x, y are relatively prime

Exercise 10. Prove that if a, b are relatively prime, then $b, a+b$ are relatively prime.

3 Proof by Cases

In certain instances where there are a small number of cases to check, you may prove a statement by dividing it up into a finite number of cases and checking that the statement holds for each separately.
To give a proof by cases, the steps are:

1. State the facts and assumptions.
2. State what is to be proven.

3. Break down all possible cases which need to be considered.

4. For each case, give a proof that concludes with the statement that needed to be shown.

Exercise 11. Show $\forall n \in \mathbb{Z}$, if n is even and $4 \leq n \leq 7$, then n can be written as a sum of two prime numbers.

Exercise 12. (Triangle Inequality) For any $x, y \in \mathbb{R}$, $|x+y| \leq |x| + |y|$.

Exercise 13. Show that if $a, b \in \mathbb{R}$, where $b \neq 0$, then

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

Exercise 14. Show that $x + |x-3| \geq 3, \forall x \in \mathbb{R}$.

Exercise 15. Show that $\forall a, b \in \mathbb{R} : ||a| - |b|| \leq |a-b|$.

4 Proof by Counterexample

Showing the statement $\forall x : (P(x) \to Q(x))$ is false is equivalent to showing that $\exists x :\sim (P(x) \to Q(x))$ which means finding an x such that $P(x)$ and $\neg Q(x)$. (You can verify this by truth table)

Remark. In order to prove a claim, giving a single example is not enough. For example, when proving that $x^2 \geq 2x - 1$ it is not enough to check that it works for one value $x = 1$. However, to disprove a universal statement, a single counterexample suffices (although sometimes there may be an infinite set of counterexamples).

Example 1. The claim “every triangle is an isosceles triangle” can be disproved using a counterexample. One example could be a 30-60-90 right triangle, but any non-isosceles triangle will similarly disprove the claim.

Remark. When proving by counterexample, you must also explain why your example is a counterexample, rather than just stating it.

Exercise 16. Explain whether or not you can disprove the following statements via counterexample:

1. All mathematicians drink coffee.

2. For any pair of real numbers x and y, $|x| + |y| = |x+y|$.

3. The polynomial $f(x) = x^3 - 3x + 2$ has a root between -1 and 0.

Solution:

1. Find a mathematician who does not drink coffee (and provide evidence that they don't).

2. Find two real numbers x and y such that $|x|+|y| \neq |x+y|$, i.e. $x=-1$ and $y=1$, so that $|x|+|y| = 1+1 = 2 \neq |x+y| = |-1+1| = |0| = 0$.

3. For this, we cannot find a single counterexample to disprove the statement. It's an existential claim, so to show that the statement is false, we would have to show that there are no roots between -1 and 0. Giving a single number $x \in (-1,0)$ that is not a root doesn't show us that there are no such x, just that that specific x does not work.

Exercise 17. Prove or disprove that $\forall a,b \in \mathbb{R} : a^2 = b^2 \implies a = b$

Definition 5. The floor function of a real number x is $\lfloor x \rfloor$, the unique integer n such that $n \leq x < n+1$.
Similarly, the ceiling function of a real number x is $\lceil x \rceil$ is the unique integer n such that $n-1 < x \leq n$

Exercise 18. Prove or disprove that $\forall x,y \in \mathbb{R}$, $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$

Exercise 19. Let x be an integer. Disprove: if x^2 is divisible by 4, then x is divisible by 4

Exercise 20. Consider real-valued functions defined on the interval $[0,1]$. We define the zero function as the function $f(x)$ which is the value zero for all values of x.
Give a counterexample to the statement: If the product of two functions is the zero function, then one of the functions is the zero function.

Exercise 21. Prove or disprove that $\forall a,b \in \mathbb{R}$,

$$(a+b)^2 = a^2 + b^2.$$

Exercise 22. Disprove that the only polynomial f for which $f(f(x)) = x$ is $f(x) = x$.

5 Solutions

Exercise 1. Prove that there is no greatest integer.

Proof. We will proceed with a proof by contradiction.
Assume $\exists$ a greatest integer N such that $N \geq n \;\; \forall n \in \mathbb{N}$. Since $m,n \in \mathbb{Z} \Rightarrow m+n \in \mathbb{Z}$, we then have $N+1 \geq N$ But this is a contradiction, so no greatest integer can exist. □

Exercise 2. Show that $\sqrt{2}$ is irrational.

Proof. Step 1: Assume the negation. $P(x) =$ "$\sqrt{2}$ is irrational" so we will assume $\sim P(x) =$ "$\sqrt{2}$ is rational."

Step 2: By definition of rational, $\exists m, n \in \mathbb{Z}$ with no common factors such that $\sqrt{2} = \frac{m}{n}$.

Step 3: To reach a contradiction, we square both sides and multiply by n^2: $\sqrt{2} = \frac{m}{n} \Rightarrow m^2 = 2n^2 \Rightarrow m^2$ is even. So, what can we say about m?

Lemma 1. x^2 is even $\Rightarrow$ x is even.

Proof. Suppose x is not even then x must be odd so there exists some integer k such that $x = 2k + 1$. Then, $x^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ so we get that x^2 is an odd number which is a contradiction.

□

Now, since we have shown that m^2 even $\Rightarrow m$ even, we must have $m = 2k$ for some $k \in \mathbb{Z}$. Then $m^2 = (2k)^2 = 4k^2 = 2n^2$ So we have $n^2 = 2k^2$, and so n^2 is even $\Rightarrow n$ is even $\Rightarrow$ both m and n have a common factor of 2, which contradicts m and n having no common factors.
Hence $\sqrt{2}$ is irrational. □

Exercise 3. Prove that there are no integers x and y such that $x^2 = 4y + 2$.

Proof. We will prove this by contradiction. Suppose $\exists x, y \in \mathbb{Z}$ such that $x^2 = 4y + 2$. But $4y + 2 = 2(2y + 1)$, and $2y + 1 \in \mathbb{Z}$, so x^2 is even.

Then since x^2 is even $\Rightarrow x$ is even (by previous Lemma), $x = 2k$ for some integer k. Then $x^2 = 4k^2$, so we have $2k^2 = 2y + 1$. But $2k^2$ is even and $2y + 1$ is odd, so they cannot be equal, and thus we have a contradiction. Therefore there are no integers x and y such that $x^2 = 4y + 2$. □

Exercise 4. The set of prime numbers is infinite.

Proof. Step 1: Let $P(x) =$ "the set of prime numbers is infinite." We will assume the negation i.e assume $\sim P(x) =$ "the set of prime numbers is finite" is true.

Step 2: If the set of all prime numbers is finite, since $\mathbb{N}$ is ordered, $\exists$ some prime number p which is the largest prime, i.e. $p \geq q, \forall$ prime numbers q, and we can write the set of prime numbers $P = \{2, 3, ..., p\}$.

Step 3: Consider the element $N = (2 \cdot 3 \cdot ... \cdot p) + 1$, so $N > p, \forall p \in P$ and $N \notin P$. Since $N > 1$ and N is not prime, then it is divisible by some prime number $q \in P$, since by the fundamental theorem of arithmetic, every number can be decomposed as a product of primes.

But each prime q divides $2 \cdot 3 \cdot ... \cdot p$, but not $(2 \cdot 3 \cdot ... \cdot p) + 1 = N$, and so this is a contradiction and N must be prime.
Hence there is no greatest prime number, and so the set of prime numbers is infinite. □

Exercise 5. Suppose $f : X \to Y$ is a function such that $f(f(x)) = x$ for all $x \in X$. Prove that f is injective.

Proof. Suppose f is not injective. Then there exists $a \neq b$ such that $f(a) = f(b)$. However, this means

$$a = f(f(a)) = f(f(b)) = b$$

which is a contradiction. Thus f is injective. □

Exercise 6. Show using the contrapositive that if the square of an integer is even, then the integer is even.

Proof. First, we state the contrapositive:

$$P(x) = \text{"the square of x is even"} \Rightarrow \sim P(x) = \text{"the square of x is odd"}$$

$$Q(x) = \text{"x is even"} \Rightarrow \sim Q(x) = \text{"x is odd"}$$

Therefore, the contrapositive statement is:

$$\forall n \in \mathbb{Z}, \text{if } n \text{ is odd} \Rightarrow n^2 \text{ is odd.}$$

Suppose n is any odd integer. Then by definition of odd, $n = 2k + 1$ for some $k \in \mathbb{Z}$.

We want to show that n^2 is odd, so we write out

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1.$$

We want this to be of the form $2m + 1$ for some m, so we rewrite this as

$$2(2k^2 + 2k) + 1.$$

Since $2k^2 + 2k$ is an integer, we set $m = 2k^2 + 2k$ and get

$$n^2 = 2m + 1.$$

so we see that n^2 is odd.

Therefore, since $(\sim Q) \Rightarrow (\sim P)$ we must have that $P \Rightarrow Q$ so we are done. □

Exercise 7. Prove that for all integers a and b, if $a + b$ is odd then a is odd or b is odd.

Proof. It is equivalent to show the contrapositive of this statement, which is that both a and b are even $\Rightarrow a + b$ is even.

Let a and b be integers, and assume both a and b are even. Then $a = 2k$ and $b = 2l$ for some integers k and l. Then $a + b = 2k + 2l = 2(k + l)$, and since $k + 1$ is an integer, we have that $a + b$ is even.

Thus we have shown $\sim$ (one of a or b is odd) $\Rightarrow$ $\sim$($a + b$ is odd) and so we have $a + b$ is odd $\Rightarrow a$ is odd or b is odd. □

Exercise 8. Prove that for every prime number p, if $p \neq 2$ then p is odd.

Proof. Let p be an arbitrary prime number, and assume p is not odd. Since every integer is either even or odd, p is even, so $p = 2k$ and p is divisible by 2.

Since p is prime, it must have exactly 2 divisors which are 1 and itself, and it has 2 as a divisor, so p must be divisible only by 1 and 2. Therefore, $p = 2$. □

Exercise 9. Prove that if x^2, y^2 are relatively prime, then x, y are relatively prime

Proof. Suppose x, y are not relatively prime. Then there exists a prime p dividing both x and y. Thus $\frac{x}{p}, \frac{y}{p}$ are integers. As an integer times an integer is an integer, $\frac{x^2}{p}, \frac{y^2}{p}$ are both integers. Thus p divides both x^2, y^2. Hence, x^2, y^2 are not relatively prime. We have now shown

$$\sim (x, y \text{ are relatively prime}) \implies \sim (x^2, y^2 \text{ are relatively prime})$$

which is exactly the contrapositive! □

Exercise 10. Prove that if a, b are relatively prime, then $b, a+b$ are relatively prime.

Proof. Suppose $a, a + b$ are not relatively prime. This means that there does exist a prime p such that p divides both b and $a + b$. Thus $\frac{a+b}{p}$ and $\frac{b}{p}$ are both integers. But as the difference of integers is an integer,

$$\frac{a+b}{p} - \frac{b}{p} = \frac{a}{p}$$

is an integer, so p divides a. Since p divides both a and b, they are not relatively prime.

We have now shown the contrapositive

$$\sim (a+b, b \text{ are relatively prime}) \implies \sim (a, b \text{ are relatively prime})$$

and so we're done. □

Exercise 11. Show $\forall n \in \mathbb{Z}$, if n is even and $4 \leq n \leq 7$, then n can be written as a sum of two prime numbers.

Proof. We only need to check for $n = 4$ and 6.
If $n = 4$ then we can write $n = 2 + 2$.
When $n = 6$ we can write $n = 3 + 3$.

□

Exercise 12. (Triangle Inequality) For any $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.

Proof. There are three cases, when x and y are both nonnegative, when x and y are both negative, and when one of x and y is nonnegative and the other is negative.

Case 1: if x and y are both nonnegative, then $(x + y)$ is nonnegative. Then $|x + y| = x + y = |x| + |y|$.

Case 2: if x and y are both negative, then $|x|+|y| = (-x)+(-y) = -(x+y)$. But since $x + y$ is also negative we get that $|x + y| = -(x + y) = |x| + |y|$.

Case 3: WLOG x is nonegative and y is negative. Then, $|x| + |y| = (x) + (-y) = x - y$.

- Case (3.a) Now, if $x \geq -y$ then $x + y \geq 0$ so $|x + y| = x + y$. So, the inequality becomes, $x - y \geq x + y$ or $-2y \geq 0$ which is true.
- Case (3.b) Now, if $x < -y$ then $x + y < 0$ so $|x + y| = -(x + y)$. So, the inequality becomes, $x - y \geq -x - y$ or $2x \geq 0$ which is true.

□

Exercise 13. Show that if $a, b \in \mathbb{R}$, where $b \neq 0$, then

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

Proof. Assume that $a, b \in \mathbb{R}$, $b \neq 0$. We want to show that

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}.$$

There are 4 cases, corresponding to $a \geq 0, a < 0, b > 0, b < 0$:
Case 1: Assume that $a \geq 0$ and $b > 0$. Then $\frac{a}{b} \geq 0$. Then $|a| = a$ and $|b| = b$ $\Rightarrow |\frac{a}{b}| = \frac{a}{b} = \frac{|a|}{|b|}$.
Case 2: Assume that $a \geq 0$ and $b < 0$. Then $\frac{a}{b} \leq 0$. Then $|a| = a$ and $|b| = -b \Rightarrow |\frac{a}{b}| = -\frac{a}{b} = \frac{a}{-b} = \frac{|a|}{|b|}$.
Case 3: Assume that $a < 0$ and $b > 0$. Then $\frac{a}{b} < 0$. Then $|a| = -a$ and $|b| = b \Rightarrow |\frac{a}{b}| = -\frac{a}{b} = \frac{-a}{b} = \frac{|a|}{|b|}$.
Case 4: Assume that $a < 0$ and $b < 0$. Then $\frac{a}{b} > 0$. Then $|a| = -a$ and $|b| = -b \Rightarrow |\frac{a}{b}| = \frac{-a}{-b} = \frac{a}{b} = \frac{|a|}{|b|}$.
Thus we have shown for all possible cases, $|\frac{a}{b}| = \frac{|a|}{|b|}$ □

Exercise 14. Show that $x + |x - 3| \geq 3, \forall x \in \mathbb{R}$.

Proof. Assume $x \in \mathbb{R}$. We want to show that $x + |x - 3| \geq 3$.
Case 1: Assume $x \geq 3$. Then $x - 3 \geq 0 \Rightarrow |x - 3| = x - 3 \Rightarrow$

$$x + |x - 3| = x + x - 3 = 2x - 3 \geq 2(3) - 3 = 6 - 3 = 3$$

Case 2: Assume $x < 3$. Then $x - 3 < 0 \Rightarrow |x - 3| = -(x - 3) = -x + 3 \Rightarrow$

$$x + |x - 3| = x + (-x + 3) = x - x + 3 = 3 \geq 3$$

So we have shown $x + |x - 3| \geq 3, \forall x \in \mathbb{R}$. □

Exercise 15. Show that $\forall a, b \in \mathbb{R} : ||a| - |b|| \leq |a - b|$.

Proof. Assume $a, b \in \mathbb{R}$. Note that if a and/or $b = 0$, then the result is trivially true, so we will only work with cases where $a, b \neq 0$:

Note that it suffices to show that the inequality holds for $a > b$. since if $a < b$ we can take $x = b$ and $y = a$ and apply the previous result for x and y.
Case 1: Assume that $a > 0$ and $b > 0$. Then $|a| = a$ and $|b| = b$, so $||a| - |b|| = |a - b| \leq |a - b|$.

Case 2: Assume that $a > 0$ and $b < 0$. Then $|a| = a$ and $|b| = -b$, so $||a| - |b|| = |a + b|$ so the inequality becomes $|a + b| \leq |a - b| = a - b$ since $a - b > 0$ or $|a + b| \leq |a| + |b|$ which is obviously true by the triangle inequality. □

Exercise 17. $\forall a, b \in \mathbb{R} : a^2 = b^2 \implies a = b$

Proof. We show the claim is false by a counterexample.
Let $a = 1$ and $b = -1$. Then $a^2 = b^2 = 1$ but $a \neq b$. □

Exercise 18. $\forall x, y \in \mathbb{R}, \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$

Proof. This is false: take $x = y = 1/2$. Then $\lfloor x + y \rfloor = \lfloor 1 \rfloor = 1$ but

$$\lfloor x \rfloor + \lfloor y \rfloor = \lfloor 1/2 \rfloor + \lfloor 1/2 \rfloor = 0 + 0 = 0 \Rightarrow \lfloor x + y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$$

□

Exercise 19. Let x be an integer. Disprove: if x^2 is divisible by 4, then x is divisible by 4

Proof. To give a counterexample, we need to find an integer x such x^2 is divisible by 4, but x is not divisible by 4.
Consider $x = 6$. Then $x^2 = 36$ is divisible by 4, but $x = 6$ is not divisible by 4. Thus, $x = 6$ is a counterexample to the statement (i.e. x satisfies $P(x)$ but not $Q(x)$).

On the other hand, consider $x = 5$. While $x^2 = 25$ is not divisible by 4, $x = 5$ is also not divisible by 4, so the "if" and "then" parts of the statement are both false. So, we cannot use $x = 5$ as a counterexample to the statement. □

Exercise 20. Consider real-valued functions defined on the interval $[0, 1]$. We define the zero function as the function $f(x)$ which is the value zero for all values of x.
Give a counterexample to the statement: If the product of two functions is the zero function, then one of the functions is the zero function.

Proof. Consider the functions

$$f(x) = \begin{cases} \frac{1}{2} - x & \text{for } 0 \leq x < \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} \leq x \leq 1 \end{cases} \qquad g(x) = \begin{cases} 0 & \text{for } 0 \leq x < \frac{1}{2} \\ \sqrt{x} + 10 & \text{for } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then multiplying these functions point-wise gives $f(x)g(x) = 0$, but neither $f(x)$ nor $g(x)$ is the zero function. □

Exercise 21. Prove or disprove that $\forall a, b \in \mathbb{R}$,

$$(a + b)^2 = a^2 + b^2.$$

Proof. Take $a = 1$ and $b = 2$. Then $(a + b)^2 = (1 + 2)^2 = 3^2 = 9$ but $a^2 + b^2 = 1^2 + 2^2 = 5$, so this is a counterexample and the claim is false.
Note: A bad proof of this would look like:

$$(a + b)^2 = a^2 + 2ab + b^2 \neq a^2 + b^2$$

While this is true in some cases, we haven't shown that $a^2 + 2ab + b^2 \neq a^2 + b^2$. Indeed, if we take either a or b (or both) to be 0, it is true. Just stating this would not be enough, so you would still need to provide a counterexample or another proof of this expanded statement. □

Exercise 22. Disprove that the only polynomial f for which $f(f(x)) = x$ is $f(x) = x$.

Proof. Take $f(x) = 1 - x$. Then

$$f(f(x)) = 1 - (1 - x) = x$$

so $f(x) = x$ is not the only polynomial solution. □