SYLLABUS FOR SYMPLECTOTHON 2024

1. Overview

Our goal is to learn about a virtual technique, using so-called global Kuranishi charts, for defining holomorphic curve invariants, following the recent papers [AMS21], [HS22] and [Hir23]. We begin with an overview of classical transversality techniques and their limitations, thereby motivating the introduction of virtual techniques. We then discuss the abstract notion of Kuranishi charts/atlases and how they give rise to well-defined virtual fundamental classes. After a detour into complex/algebraic geometry, we discuss the construction of global Kuranishi charts for Gromov-Witten moduli spaces. Finally, we discuss how global Kuranishi charts allow us to give clean proofs of the Kontsevich–Manin axioms for Gromov–Witten invariants.

2. List of talks

2.1. **Overview and motivation.** Give a high-level overview of how 'curve counting invariants' are defined. Start from 'classical' methods, motivate the need for virtual techniques, and explain why we care to define invariants in the most general setting.

Topics to cover.

- Generic transversality for simple curves [MS12, Ch 3] and why this allows us to define GW invariants in the semipositive case using pseudocycles [MS12, Ch 6].
- Examples to show that where transversality for generic J fails in general for multiple covers. E.g., branched multiple covers in Calabi–Yau 3-folds [Wen23, Ex 1.2].
- Examples where transversality fails but we still have an obstruction bundle. E.g., [MS12, Thm 7.2.3] or Aspinwall–Morrison formula [HKK$^+$03, §27.5].
- High-level overview of virtual fundamental classes, e.g. following [HKK$^+$03, §26.1].
- Why we want invariants to be defined in general (and not just semipositive) case? Explain using your favorite example. E.g., Fukaya–Ono's (and others') generalization of Floer's proof of Arnol'd conjecture, using Kuranishi structures (and their variants).

2.2. **Orbifolds as groupoids.** Discuss the definition of orbifolds in terms of groupoids, and what advantages this perspective offers over the "classical" definition. Give some simple examples of orbifolds and how they tend to arise, and related concepts such as morphisms between orbifolds.

Topics to cover.

- Some of the ideas discused in Wendl's recent blog post [Wen].
- See also the survey article [Moe02].
- You may also wish th consult the classical textbook reference [ALR07].

2.3. **Local Kuranishi charts and atlases.** Sketch the definition of a Kuranishi atlas (or a variant) and how it is used to define a virtual fundamental class. Explain why holomorphic curve moduli spaces carry such an atlas.

Topics to cover.

- Abstract definition of Kuranishi (or implicit) atlases [Par16, §2.1.1 & §3.1].
- Finite dimensional reduction, i.e., implicit function theorem for Fredholm operator but not necessarily surjective linearization [BS21, Appx A].
- Constructing such an atlas on the zero locus of a Fredholm section of a Banach bundle, adapting discussion in [Par16, §2.2.1 & §5.3].
- Constructing such an atlas on a holomorphic curve moduli space [Par16, §2.2.2 & §9.2].
- How to use an atlas to construct a virtual fundamental class [Par16, §2.3.1–2.3.3]. Optional: why the case of two charts is already quite subtle [Par16, §2.3.4].

2.4. **Global Kuranishi charts.** Define the notion of a global Kuranishi chart, equivalence, cobordism and orientation. Define the virtual fundamental class determined by an oriented global Kuranishi chart and prove its invariance under equivalence.

Topics to cover.

- Definition of global Kuranishi charts (GKC) and notions of orientation, equivalence and cobordism for GKC; [AMS21, §4.1] or [HS22, §2.3].
- Definition of virtual fundamental class (VFC) using an oriented GKC; [AMS21, Eq (5.1)] or [HS22, Def 2.28]. Why we need $\mathbb{Q}$ coefficients (Poincaré duality for orbifolds).
- Invariance of VFC under equivalence of GKCs [HS22, Lem 2.29].
- Discuss examples of inequivalent GKCs on the same compact Hausdorff space. E.g., zero locus of $\mathfrak{s}_n : \mathbb{C} \to \mathbb{C}$ given by $z \mapsto z^n$ for different choices of $n \geq 1$, or constant stable maps of genus $g \geq 2$ to a closed symplectic manifold which is not a point.

2.5. **Linear systems and peak sections.** Discuss linear systems and ampleness. Introduce "framed curves" and discuss how to achieve transversality for them using "Hörmander peak section" perturbations.

Topics to cover.

- Definition of linear systems and what it means to be basepoint free, ample and very ample [Har77, §II.7]. Statement of the Serre vanishing theorem [Har77, Thm III.5.2].
- Correspondence between basepoint free linear systems and maps to $\mathbb{P}^n$ [Har77, §II.7].
- Numerical criterion for ampleness on curves [Har77, Prop IV.3.1 & Cor IV.3.2].
- Definition of "framed curves"; [AMS21, Def 6.10] or [HS22, Def 2.4]. Why "Hörmander peak section" perturbations achieve transversality for framed curves [AMS21, §6.5].

2.6. **Moduli spaces of curves in $\mathbb{P}^n$.** Discuss moduli spaces of holomorphic curves in $\mathbb{P}^n$, focusing on the cases where they are complex orbifolds.

Topics to cover.

- Vanishing of $H^1(C, \varphi^*\mathcal{O}_{\mathbb{P}^n}(1))$ implies Fredholm regularity of a holomorphic curve $\varphi\colon C \to \mathbb{P}^n$ [HS22, Lem 3.2]. Deduce corollary: $\overline{\mathcal{M}}_{0,m}(\mathbb{P}^n, d)$ is a complex orbifold.
- Algebraic GKC for rational curves in a hypersurface in $\mathbb{P}^n$ [HKK$^+$03, §26.1.3].
- Global quotient presentation for $\overline{\mathcal{M}}_g$ via the Hilbert scheme of tri-canonically embedded curves in $\mathbb{P}^{5g-6}$ modulo $\mathrm{PGL}(5g-5)$; [DM69, pages 77–78] or [ACG11, §5].
- Optional: explicit description of a low degree moduli space of rational curves in $\mathbb{P}^n$. E.g., describing $\overline{\mathcal{M}}_0(\mathbb{P}^2, 2)$ as a blow up of $\mathbb{P}^5$.

2.7. **Global Kuranishi charts for genus 0 GW theory.** Explain the construction of GKCs for genus 0 GW moduli spaces in [AMS21] and their uniqueness up to equivalence.

Topics to cover.

- Construction of GKC given in [AMS21, §6]; see [HS22, §2.1] for a summary.
- Construction of the doubly-thickened GKC in [AMS21, §6.10] or [HS22, §6.1] and how this yields equivalence of GKCs obtained by making different choices.
- Optional: the GKC is oriented/stably complex [HS22, §5.5].

2.8. **Adapting the construction to higher genus holomorphic curves.** Explain why the genus 0 construction from [AMS21] does not work for higher genus. Explain how to overcome these issues, following [HS22].

Topics to cover.

- Difficulties with generalising the construction to higher genus, [HS22, Discussion 2.10].
- Picard group of a higher genus (nodal) curve [HS22, Appx B],
- Construction of higher genus GKC [HS22, Construction 3.13]. Explanation of how the α-component of the obstruction section resolves the issues mentioned in [HS22, Discussion 2.10].

2.9. **Kontsevich–Manin axioms for GW theory.** State the Kontsevich–Manin axioms and the moral reason why they are true. Illustrate how global Kuranishi charts allow us to turn moral arguments into rigorous arguments.

Topics to cover.

- Statement of the axioms (maybe only in genus 0) and motivation behind them [KM94].
- Explanation for why we can arrange for the evaluation maps on the thickening to be submersions but not for the stabilization map.
- Proof sketch of the Splitting axiom in genus 0 [Hir23, §3.4].

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