Symplectothon 2024

Global Kuranishi charts

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1 Saturday

1.1 Kyler Siegel - Overview

1.1.1 Motivation

This is a technical meeting, this talk is for motivation. The topic of this is Global Kuransihi Charts, an important emerging topic in Symplectic Geometry. There are many invariants in symplectic geometry, but in many of them there is the technical issue fo defining virtual fundamental classes. GLobal Kuranishi Charts allows us to do this in a more streamlined way. Here's a result first proven using GKC that would have been difficult without them:

Theorem 1: AMS '21

Let M^{2n} be a closed symplectic manifold, $\pi : M \to s^2$ be a smooth submersion whose fibers are sympletic submanifolds of M. Then $H^*(M;\mathbb{Z}) \cong H^*(S^2,\mathbb{Z}) \otimes H^2(\text{fiber},\mathbb{Z})$ (as graded abelian groups)

This was known much earlier for M monotone (Lalonde-McDuff-Polterovich), and over $\mathbb{Q}$ for any symplectic manifold (McDuff). The hypothesis of this theorem automatically hold if M is a smooth complex projective variety and $\pi : M \to \mathbb{P}^1$ is any morphism (this was proved by Deligne using algebraic methods). There's a version of this theorem using more general cohomology theories. This was proven using Gromov-Witten Invaraints

1.1.2 Gromov-Witten Invariants

Given any closed symplectic manifold, M^{2n} and $A \in H_2(M,\mathbb{Z})$ with $\{\beta_i\}_{i=1}^k \in H_*(M)$, and $g \geq 0$, then we should be able to define

$$GW_{M,A,g}\langle \beta_1, ..., \beta_k\rangle \in \mathbb{Q}$$

Which are symplectomorphism invariants of M. The rough idea for defining these things is as follows. Start by choosing an almost complex structure J compatible with the symplectic form. Then for a Riemann surface (Σ, j_Σ)

$$GW_{M,A,g}\langle \beta_1, ..., \beta_k\rangle = \#\{u : \sigma \to M | \overline{\partial} u = Jdu - du \circ j_\Sigma = 0\}/\sim$$

With $z_1, .., z_k \in \Sigma$ and $u(z_i) \in \beta_i$ for all i, and $[u] = A$. The identification is give by identifying $u : \Sigma \to M$ and $u' : \Sigma' \to M$ if there is a biholomorphism $\phi : \Sigma \to \Sigma'$ such that $u' \circ \phi = u$ and $\phi(z_i) = z_i'$, that is map marked points to marked points. This has initial issues:

1. Why is this a finite count?
2. Why is this independent of our choice of almost complex structure J

Both of these *should* be solved by Gromov's compactness arguments. Consider

$$\widetilde{\mathcal{M}}^J_{M,A,g,k} = \{u : \Sigma \to M | \overline{\partial} u = 0\}$$

and then define $M^J_{M,A,g,k} = \widetilde{\mathcal{M}}^J_{M,A,g,k}/\sim$. now, let $\overline{\mathcal{M}^J}$ be the compactification of $\mathcal{M}^J$ by stable maps. The idea here is to allows nodal Riemann Surfaces Σ which are "stable". The surface is stable if any compontent of Σ on which u is constant with $g = 0$ must have ≥ 3 special points, and when $g \geq 1$ you need ≥ 1 special points. If we knew that $\overline{\mathcal{M}}^J$ was a smooth manifold, so an oriented closed manifold then we could reformulate these GW-invariantts as

$$GW_{M,A,g}\langle\beta_1, ..., \beta_k\rangle = \int_{[\overline{\mathcal{M}}^J]} \mathrm{ev}_i^* PD(\beta_1) \smile ... \smile \mathrm{ev}_k^* PD(\beta_k) \in \mathbb{Q}$$

Where $\mathrm{ev}_i : \overline{\mathcal{M}}J \to M$ takes a curved to the image of the i-th marked point. Then $[\overline{\mathcal{M}}^J] \in H_{\text{top}}[\overline{\mathcal{M}^J}]$. Unfortunately, it is usually not the case that these are smooth manifolds. There is one nice thing to say however. Here's a classical fact(90s): If we choose a generic complex structure $J \in \mathcal{J}(M)$, then the subspace of simple curves in $\mathcal{M}^J$ is a smooth manifold of dimension $(n-3)(2-2g) + 2c_1(A) + 2k$ where M is of diemsnion $2n$

Definition 1: Simple J-Holomorphic Curves

A J-holomorphic curve $u : \Sigma \to M$ is a "multiple cover" if $f : \Sigma \to \Sigma'$ is a holomorphic function of degree at least 2, and $r : \Sigma' \to M$ such that $u = r \circ f$. Otherwise, we call u "simple"

If every curve was simple we'd be done. However the multiple covers usually give us trouble as near them $\overline{\mathcal{M}}$ is either singular or of the wrong dimension. Let's see a simple example

Example 1.1. Suppose M is a Calabi-Yau 3-manifold, so $n = 3$ and $c_1(M) = 0$, and assume we have no marked points. Then

$$\mathrm{ind}\overline{\mathcal{M}}_{M,A,g,0} = 0$$

Given $u : \Sigma \to M$, given a branched cover $f : \Sigma' \to \Sigma$ we get $u \circ f$ another J-holomorphic curve. There are a lot of such maps, and we get a large dimension worth of them when we expected a zero diemsnional one. Suppose we look at deg κ maps $\mathcal{P}^1 \to \mathcal{P}^1$. Then

$$\dim\{\mathbb{P}^1 \to \mathbb{P}^1\}/\sim = 2(1-3) + 4\kappa = 4\kappa - 4$$

If we set $\mathcal{B} = \{u : \Sigma \to M\}$, then there's a (banach) vector bundle $\mathcal{E} \to \mathcal{B}$ with fibers

$$\mathcal{E}_u = \Gamma^{0,1}(u^*TM) = \mathrm{Hom}^{0,1}(T\Sigma, u^*TM)$$

There's a section $\overline{\partial} \in \Gamma(\mathcal{E})$ such that $\widetilde{\mathcal{M}}$ is just the zero section

Definition 2: Regular Curve

We say that u is "regular" if the linearization of $\overline{\partial}$ at u, Du is surjective.

Fact: $\widetilde{\mathcal{M}}$ is smooth near regular curves. The best thing you could hope for is that the multiple covers are an orbifold.

Definition 3: Semipositive Symplectic Manifold

A symplectic manifold (M, ω) is "semi-positive" if for all $A \in \pi_2(M)$ such that $\omega(A) > 0$ and $c_1(A) \geq 3 - n$, then $c_1(A) \geq 0$

Theorem 2: Ruan-Tian, McDuff-Salamon

For generic almost complex structure $J \in \mathcal{J}(M)$,

$$\text{ev} : \mathcal{M}^{J,\text{simp}}_{M,A,g,k} \to M^{\times k}$$

is a psuedo-cycle of dimension $(n-3)(2-2g) + 2c_1(A) + 2k$.

In this case, $[\overline{\mathcal{M}}^J]$ has a $\mathbb{Z}$-valued fundamental class. This is about as far as you can get with classical methods, as the multiple covers *really* start to cause problems. As a fix, we can define *virtual* fundamental classes $[\overline{\mathcal{M}}^J]^{\text{vir}} \in H_{(n-3)(2-2g)+2c_1(A)+2k}(\overline{\mathcal{M}}^J, \mathbb{Q})$. When would you have a clear expectation of what this virtual fundamental class may be?

Example 1.2. Suppose $\overline{\mathcal{M}}^J$ actually is a smooth manifold of the wrong dimension. Further assume that for each $u : \Sigma \to M \in \overline{\mathcal{M}}^J$, $T_u\overline{\mathcal{M}}^J = \ker Du$, but not that Du is surjective. We can then define a vector bundle $E \to \overline{\mathcal{M}}^J$ such that the fiber $E_u = \text{coker} Du$. Then $\dim \overline{\mathcal{M}}^J - \text{rank}(E) - \text{virtdim}(\overline{\mathcal{M}}^J)$.

Definition 4: Virtual Fundamental Class

$[\overline{\mathcal{M}}^J]^{\text{vir}} = e(E) \cap [\overline{\mathcal{M}}^J] \in H(\overline{\mathcal{M}}^J)$

Patching these together globally can be troublesome, which is what Global Kuranishi Charts set out to help solve.

1.2 Roman Krutowski- Groupoids and Orbifolds

1.2.1 Topological/Lie Groupoids

Definition 5

A groupoid is an essentially small category with obejectis X and morphisms G, denoted $X \rightrightarrows X$ with all arrows invetible, along with structure maps (s, t, c, u, i)

$$\begin{aligned} s, t &: G \to X \\ c &: G(y,z) \times G(x,y) \to G(x,z) \\ u &: X \to G \\ x &\mapsto e_x \\ i &: G \to G \\ g &\mapsto g^{-1} \end{aligned}$$

Recall that essentially small means that all homs $G(x, y)$ are sets, and there is a set $S \subset X$ such that for all $x \in X$, there is an equivalent element to x in S. Any group is groupoid with a single object, but there are more interesting examples:

1. If H acts on X we get a groupoid $G \rightrightarrows X$ where
$$G = \{(x, h, y) \in X \times H \times X | hx = y\}$$
2. The category of genus G surfaces with m marked points, $\mathcal{D}_{g,m}$ where the morphisms are diffeomorphisms
3. The path groupoid. If X is a topological spaces, then $\mathcal{P} \rightrightarrows X$ where $\mathcal{P} = \bigsqcup_{(x,y)\in X\times X} \mathcal{P}(x, y)$ where $\mathcal{P}(x, y)$ are homotopy classes of paths from x to y

Definition 6: Topological Groupoids

A "Topological Groupoid" is small with (G, x)-spaces and structural maps continuous. We denote by $|X| = X/\sim$ where we identify points with an arrow betwen them.

Definition 7: Open and Proper Groupoids

We say that a topological groupoid $G \rightrightarrows X$ is "open" is s, t are open maps, and "proper" if s, t are proper maps

Theorem 3: Hausdorff Quotients

$|X|$ is Hausdorff for open, proper topological groupoid with Huasdorff X

Definition 8: Proper Lie Groupoid

We say that $G \rightrightarrows X$ is a "proper Lie Groupoid" if G, X are smooth manifolds with structure maps, and s, t are in addition submersions

Theorem 4

For $G \rightrightarrows X$ a proper Lie Groupoid, and $x, y \in X$, then $G(x, y)$ is a closed submanifold of G, G_x is a Lie group, Gx is an immersed submanifold of X, and $t_x : G(x, X) \to Gx$ is a G_x-principal bundle.

Proof. Here's a sketch: Note that $G(x, X) = s^{-1}(x)$ is a manifold as s is a submersion. Define $E_g = \ker(ds)_g \cap \ker(dt)_g \subset T_g G$. We claim that $E_g|_{G(x,X)}$ is an involutive subbundle. Now, consider $L_g : G(X, x) \to G(x, t(g))$ which is a diffeomorphism and $(dL_g)(E_{1_X}) = E_g$ giving us the claim. Thus we have a foliation, and so leaves of $E_g|_{G(x,X)}$ are connected components of the fibers of t_x □

1.2.2 Slices

Definition 9

Let $f : Y \to X$ be a map, then we can associate a groupoid pullback of $G \rightrightarrows X$, $f^*G \rightrightarrows Y$ by

$$f^*G(y_1, y_2) = G(f(y_1), f(y_2))$$

Proposition 1

Suppose $G \rightrightarrows M$ is a proper Lie Groupoid, and $f : N \to M$ is a transverse to all G-orbits. Then

1. $f^*G \rightrightarrows N$ is naturally a proper Lie Groupoid
2. $|N| \to |M|$ is a homeomorphism onto it's image

We can see a special case of this in slices

Definition 10: Slices

A submanifold $S \subset M$ is a "slice" through $x \in M$ if $T_x M = T_x S \oplus T_x(Gx)$

Note that this is the same as $i^*G = G(S, S) \to S$ as a local groupoid. Given a collection of slices $\{S_\alpha\}_{\alpha \in A}$, then let $S = \bigsqcup_{\alpha \in A} S_\alpha$ with $i_\alpha : S \to M$ which induces $i^*G \to S$. These give you local topology of a quotient

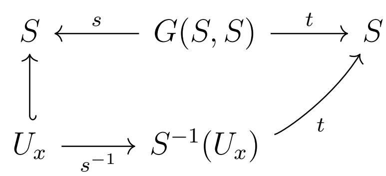
Definition 11: Weak Equivalence of Lie groupoids

A map(functor) $f : (H \rightrightarrows N) \to (G \rightrightarrows M)$ is a "weak equivalence" if in addition to preserving the smooth structures, $f : N \to M$ has nonempty transverse intersections with every orbit, and $H(x,y) \cong G(f(x), f(y))$

Definition 12: étale Lie Groupoid

We say that a Lie Groupoid is "étale" if s and t are local diffeomorphisms

Now let $g \in G_x$. We'd like an action of g on S. If our groupoid is étale, then we have diffeomorphisms



So we can define $\phi_g = t \circ s^{-1}$. For a small neighborhood, we get V_x with a smooth action of G_x on V_x giving is $|V_x| \cong V_x/G_x$. This is exactly what the orbifold picture says, these are the local uniformizers where we have a local chart modded out by a finite group action.

1.2.3 Large Groupoids

Main example of why we may need Large Groupoids is as follows. Fix J an almost complex strcutre on a symplectic manifold M. Define

$$\mathcal{E}_J \rightrightarrows X_J$$

Where

$$X_J = \bigsqcup_{(\Sigma,\theta)\in\mathcal{D}_{g,m})} \widetilde{\mathcal{M}}(\Sigma,\theta,M)$$

$$\mathcal{E}_J = \bigsqcup_{((\Sigma,\theta),(\Sigma',\theta'))\in\mathcal{D}_{g,m}\times\mathcal{D}_{g,m}} \mathcal{E}((\Sigma,\theta),(\Sigma',\theta'))$$

for $u_0 = (\Sigma, j_0, \theta, u_0)$ we can associate a chart

$$U = \{(\Sigma, j, \theta, u) | (j,u) \in \widetilde{\mathcal{M}}\}$$

Proposition 2

let $\{U_\alpha \subset X\}_{\alpha \in I}$ be a collection of local models, then for

$$G_I = \bigsqcup G(U_\alpha, U_\beta) \rightrightarrows U_I := \bigsqcup U_\alpha$$

We get $i : |U_I| \to |X|$ a homeomorphism onto it's image.

1.3 Austin Christian- Local Kuranishi Charts and Implicit Atlases

1.3.1 Implicit Atlases

An implicit atlas ought to exist on zero sets, whether or not 0 is a regular value or not. Imagine you have a smooth function $f : Y \to \mathbb{R}$ for Y a smooth manifold, and consider $X = f^{-1}(0)$. We'd like to talk about X whether 0 is a regular value or not, so consider

$$\begin{aligned} F : Y \times \mathbb{R} &\to \mathbb{R} \\ (y, z) &\mapsto f(y) + z \end{aligned}$$

For which $F^{-1}(0)$ is smoothly cut out and $X \subset F^{-1}(0)$. Projecting $Y \times \mathbb{R} \to \mathbb{R}$ cuts out X. To get the definition, replace $Y \times \mathbb{R}$ with an arbitrary vector bundle $p : E \to B$ where $X := s^{-1}(0)$ for some smooth section $s : B \to E$. Locally, we'd like X to look like the zero set of projection to a fiber restricted to a submanifold

Definition 13: Proto-Implicit Atlas

For a compact Hausdorff space X, a "proto-implicit atlas" consists of an index set A and the following data:

1. A finite dimensional vector space E_α for each $\alpha \in A$ called the "obstruction spaces", and note for $I \subset A$, $E_I := \bigoplus_{\alpha\in I} E_\alpha$

2. A topological Manifold X_α called the "thickenings" for each $\alpha \in A$. In fact, X_I for each $I \subset A$ finite, with $X_\emptyset$ is homeomorphic to X

3. A map $s_\alpha : X_I \to E_\alpha$ called a "Kuranishi map" for each $\alpha \in I$, and for each $I \subset J$ we define $S_I : X_J \to E_I$ as $s_I = \bigoplus_{\alpha\in I} s_\alpha$

4. The Kuranshi maps satisfy a "transversality axiom": For each $\emptyset \neq I \subset J \subset A$ finite, the map $s_{J\setminus I} : X_J \to E_{J\setminus I}$ is locally on the projection

   $$\mathbb{R}^{d+\dim(E_J)} \to \mathbb{R}^{\dim(E_{J\setminus I})}$$

   over $o \in E_{J\setminus I}$

5. An open set $U_{IJ} \subset X_I$ for all $I \subset J \subset A$ finite called a "footprint" such that

   (a) $X_\emptyset = \bigcup_{\emptyset\neq I\subset A \text{ finite}} U_{\emptyset I}$
   (b) $U_{IJ} \cap U_{IK} = U_{I,J\cup K}$
   (c) $U_{II} = X_I$

6. For each $I \subset J \subset A$ finite, a homeomorphism called a "footprint map"

   $$\psi_{IJ} : (s_{J\setminus I}|_{X_J})^{-1}(0) \to U_{IJ}$$

   Satisfying:

   (a) $\psi_{IJ} \circ \psi_{JK} = \psi_{IK}$ with $\psi_{II} = \text{id}$
   (b) $s_I \psi_{IJ} = s_I$

   (c) $\psi_{IJ}^{-1}(U_{IK}) = U_{JK} \cap (s_{J\setminus I}|_{X_J})^{-1}(0)$

So what is a Local Kuranishi chart? Each $\alpha \in A$ gives us a basic chart or a "local Kuranishi Chart". For our space X, we've included an open subset $s_\alpha^{-1}(0)$ which also includes into X_α as a closed set:

$$X \xleftarrow{\text{open}} s_\alpha^{-1}(0) \xrightarrow{\text{closed}} X_\alpha \xrightarrow{s_\alpha} E_\alpha$$

1.3.2 Zero Locus of a Banach Bundle

Let $p : E \to B$ be a vector bundle with a section $s : B \to E$. We want to build an implicit atlas on $s^{-1}(0)$. Let A be all possible thickening data where a thickening datum consists of

- An open set $V_\alpha \subset B$
- A finite dimensional vector space E_α
- A smooth homomorphism
$$\lambda_\alpha : V_\alpha \times E_\alpha \to p^{-1}(V_\alpha)$$

For each $I \subset A$ finite, we define a thickening as

$$X_I = \left\{ (x, [e]_{\alpha \in I}) \middle| x \in \bigcap_{\alpha \in I} V_\alpha, s(x) + \sum_{\alpha \in I} \lambda_\alpha(x, e_\alpha) = 0 \right\} \subset V_I \times E_I$$

Where $V_I = \bigcap_{\alpha \in I} V_\alpha$. The Kuranishi maps $s_\alpha : X_I \to E_\alpha$ are projections, with footprints

$$U_{II} = \{(x, [e]_{\alpha \in I}) | x \in V_I\} \subset X_I$$

and footprint maps

$$\psi_{IJ}(x, [e_\alpha]_{\alpha \in J}) = (x, [e_\alpha]_{\alpha \in I})$$

1.4 Soham Chanda - Global Kuranishi Charts

What is a Global Kuranishi chart? It's additional structure given to a compact Hausdorff Z, so start with a vector bundle $E \to T$ over a smooth manifold T wish a section s

So at this point we have $(-, E, T, s)$ representing Z. This missing piece of data her is a compact lie group G which acts on E (linearly on fibers), acts on T with finite stabilizers, and s is G-equivariant. This 4-tuple is a Global Kuranishi chart, or GKC, for Z, so $Z \cong s^{-1}(0)/G$. Noe that

$$\text{vdim}_k(Z) = \dim T - \dim G - \text{rank}(E)$$

Note that homeomorphic spaces can have different GKCs.

Example 1.3. Consider the GKCs $(0, C \times \mathbb{C}, \mathbb{C}, (z \mapsto (z, z^2)))$ where $s^{-1}(0) = \{0\}$ so we have $\text{vdim}(Z) = 0$. We can also have $(0, \mathbb{C} \times \mathbb{C}^2, \mathbb{C}, (z \mapsto (z, z^2, z^2)))$, for which $s^{-1}(0) = \{0\}$ and $\text{vdim}(Z) = -1$

Note that we need both T and E to be oriented, and for G to be an orientation preserving action for this to be an oriented GKC.

1.4.1 Equivalence of GKCs

There are a few ways of talking about equivalence

1. Germ equivalence: Given $s^{-1}(0) \subset U \subset T$ we look at $(G, E|_U, U, s|_U)$ and we can consider equivalence here.

2. Next, if we have a vector bundle $p : W \to T$ for which G acts nicely, then we can consider $p^*W \oplus p^*E \to W$ which comes with a section $s_W = \Delta_W \oplus p^*s : W \to p^*W \oplus p^*E$. The data of $(G, p^*W \oplus p^*E, W, s_W)$ will give a "stabilization equivalence" GKCs.

3. Another choice here is that you can start with a G that is "too big", and see what happens. Given a Q equivariant principal G'-bundle $q : P \to T$, we can consider the GKC
$$(G \times G', q^*E, P, q^*s)$$
Which is equivalent to (G, E, T, s) in some sense.

In the second equivalence, we have that $\Delta_w(w) = 0$ implies that $w \in T_0$ and so $w \in s^{-1}(0)$.

1.4.2 Virtual Fundamental Class

A virtual fundamental class is a map $\check{H}^d(Z, \mathbb{Q}) \to \mathbb{Q}$ where $d = \text{vdim}(Z)$. We have to build this. Starting off, the Thom class of a vector bundle $E \to T$ or rank r is $\text{Th} \in H^r(E, E \setminus T)$ such that $\text{Th}|_p := \iota_p^* T$ is a generator of $H^r(E_p, E_p \setminus 0)$. Now, given $\check{H}^d(Z, \mathbb{Q}) = \lim_{\to} H^d(U, \mathbb{Q})$ where $U \subset Z$ is a neighborhood in T/G. Given $s : (T/G, T/G \setminus Z) \hookrightarrow (E/G, E/G \setminus T/G)$ we can look at
$$H^*(U, \mathbb{Q}) \xrightarrow{s^*\text{Th}_{E/G} \smile} H^*(U, U \setminus Z) \to H_c^*(T/G, \mathbb{Q})$$
We can then define the VFC as
$$\begin{array}{ccc} \check{H}^d(Z, \mathbb{Q}) & \xrightarrow{s^*\text{Th}_{E/G} \smile} & H_c^{\dim(T/G)}(T/G, \mathbb{Q}) \\ & \searrow^{\text{VFC}} & \downarrow [T/G] \\ & & \mathbb{Q} \end{array}$$

> **Theorem 5**
>
> VFC is invariant under equivalent GKC

Example 1.4. Let $K_1 = (0, \mathbb{C} \times \mathbb{C}, \mathbb{C}, s_1 = (z \mapsto z^2))$ and $K_2 = (0, \mathbb{C} \times \mathbb{C}, \mathbb{C}, s_2 = (z \mapsto z^3))$. Here $Z = s^{-1}(0) = \{0\}$, so we get a map for K_1
$$\check{H}^0(*) \xrightarrow{VFC_{K_1}} \mathbb{Q}$$
$$* \mapsto 2$$

Doing the same with K_2 gives 3 in the image of this map, so these GKCs are not equivalent.

1.5 Siyang Liu - Everything BUT Kuranishi Charts

1.5.1 Bundles and Sections

The goal here is to describe geometric foundations to construct GKC on the moduli space of psuedo-holomorphic curves. Let X be a smooth projective variety, and L be a holomorphic line bundle on X. To this bundle we can associate $D \subset X$ a hypersurface (Divisor) as follows; given $s \in H^0(X, L)$ we can consider $s^{-1}(0) \subset X$ which is out divisor. Note that $s^{-1}(0) = (\lambda s)^{-1}(0)$ for all $\lambda \in \mathbb{C}$. In other words, we get a "linear system" $|H^0(X, L)| = \mathbb{P}(H^0(X, L))$

Proposition 3

Given a divisor $D_0 \subset X$, and let $L = L(D_0)$ that is the sections vanishing along D_0, then there is a one to one correspondence

$$\{D \text{ effective, up to linear equivalence}\} \leftrightarrow \{s \in H^0(X, L) | D = s^{-1}(0)\}$$

Definition 14

L (or $|H^0(X, L)|$) is "base-point free" if $\nexists p \in X$ such that $p \in D$ for all $D \in |H^0(X, L)|$

Definition 15: (Very) Ample

L is ample if $\exists m >> 0$ such that $L^{\otimes m}$ is base-point free. If in addition L separates points and tangent vectors, then we say that L is "very ample"

Theorem 6

L is very ample if and only if there exists an embedding $\varphi : X \hookrightarrow \mathbb{P}^N$ such that $\varphi^* \mathcal{O}_{\mathbb{P}^N}(1) = L$.

The idea here is if we're base-point free, then $\{s_i\}$ is a basis of $H^0(X, L)$ so $[s_0 : ... : s_k] : X \to \mathbb{P}^k$ is well-defined

Theorem 7: Serre

If X is projective over A noetherian, $\mathcal{O}_X(1)$ is a very ample line bundle, and $\mathcal{F}$ is a coherent sheaf, then

1. $\forall i \geq 0$, $H^i(X, \mathcal{F})$ is a finitely generated A-module
2. $\exists n_0$ depending on $\mathcal{F}$ such that $\forall i > 0$ and $\forall n \geq n_0$, we have

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes n}) = 0$$

1.5.2 Framed Curves

Looking at $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$ where d is the degree of the curve, then

$$H^*(\mathbb{P}^d, \mathbb{Z}) \cong \mathbb{Z}[h]/h^{d+1}$$

and $h \in H^2(\mathbb{P}^d, \mathbb{Z})$ such that $h(u_*[\Sigma]) = d$. This moduli space is typically an orbifold. moreover we have $\mathcal{F} \subset \mathcal{M}$ consisting of nodal holomorphic spheres not contained in any hyperplanes $\{x_i = 0\}$ for $0 \leq i \leq d$. We also have a "universal curve" $\mathcal{C} \to \mathcal{F}$ whose fiber at each point is the curve itself

Proposition 4: AMS '21

Both $\mathcal{F}$ and $\mathcal{C}$ are quasi-projective and smooth

Proof. Given $u \in \mathcal{F}$, we have a map $u : \Sigma \to \mathbb{P}^d$. This gives us a bundle $u^*\mathcal{O}_{\mathbb{P}^d}(1)$ which is a very ample line bundle over Σ with a prescribed basis $\{u^* x_i\}$ of $H^0(\Sigma, u^*\mathcal{O}_{\mathbb{P}^d}(1)$. So we have a bijection

$$\mathcal{F} \leftrightarrow \{(\Sigma, L, F) |\deg L = d\}/\mathrm{Aut}$$

Where the automorphisms are maps $\phi : \Sigma \to \Sigma$ lifting to $\phi : L \to L$ such that $\phi^* : H^0(\Sigma, L) \to H^0(\Sigma, L)$ is the identity. The smoothness corresponds to the fact that the automorphism group is trivial. □

Definition 16: Framed Curve

A "framed curve" is a triple $(\Sigma, L, \mathcal{F})$ where L is a very ample line bundle and $\mathcal{F}$ is a basis of $H^0(\Sigma, L)$ such that

1. Σ has $g = 0$
2. L should be strictly positive on any unstable component of Σ
3. $F = \{s_1, ..., s_d\}$ a basis of $H^0(\Sigma, L)$ has

$$\mathcal{H}(\Sigma, L, F) := \left(\int_\Sigma \langle s_i, s_j \rangle u^* \Omega \right)_{i,j}$$

   Has positive eigenvalues

1.5.3 Hörmander's Peak Sections

Fix a Kähler manifold (Y, J, ω) and a vector bundle $E \to Y$ and and consider the map $- \wedge \omega : \Omega^*(E) \to \Omega^*(E)$ and the metric adjoint of this is called $\Lambda_\omega : \bigwedge^{p,q} T^*Y \otimes E \to \bigwedge^{p-1,q-1} T^*Y \otimes E$

Lemma 1

Fix an effective divisor $D \subset Y$ and $x \in Y \setminus D$, and let δ_e be the dirac delta at X with value $e \in E$. Then there exists $s_k, \check{s}_k$ of $L^{\otimes k}$ and $E \otimes L^{\otimes k}$ such that $\langle s_k, \check{s}_k \rangle \to \delta_e$ as $k \to +\infty$, so $s_k, \check{s}_k$ vanish along D

2 Sunday

2.1 Julian Chaidez- Moduli Spaces in $\mathbb{P}^n$

We've talked a lot about virtual fundamental cycles and Kuranishi charts, the point of this talk is to see specific examples and check things directly.

2.1.1 Recollection

$\overline{\mathcal{M}}_{g,n}(X^{2m}, A)$ is a set of maps $u : \Sigma_g \to X$ and some fixed marked points $z_1, .., z_n \in \Sigma_g$ such that u is J-holomorphic ands table, and then we quotient this set out. Recall that if every curve in $\overline{\mathcal{M}}_{g,n}(X, A)$ is regular (Du surjective), then the moduli space is an aorbifold of dimension

$$\text{vdim}(\overline{\mathcal{M}}_{g,n}(X^{2m}, A)) = 2c_1(TX)A + 2(m-3)(1-g) + 2n$$

Note that for $u : (\Sigma, j) \to (X, J)$ and J integrable, u is regular if and only if

$$H^1(\Sigma, u^*TX) = 0$$

One way to see this is that the cohomology of u^*TX can be computed as the cohomology of

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^\infty(\Sigma, u^*TX) & \xrightarrow{Du} & C^\infty(\Sigma, \overline{T^*\Sigma} \otimes_{\mathbb{C}} u^*TX) & \longrightarrow & 0 \\ & & \Big| & & \Big| & & \\ \dots & \longrightarrow & \Omega^{0,0}(u^*TX) & \longrightarrow & \Omega^{0,1}(u^*TX) & \longrightarrow & \dots \end{array}$$

The point of this is that Du is onto if and only if $H^1(u^*TX) = 0$.

2.1.2 Moduli of Curves in $\mathbb{P}^n$

Let's start by looking at

$$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$$

We claim that every curve is regular in this moduli space

Lemma 2

$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$ is an orbifold of complex dimension

$$d(n+1) + (n-3) + n$$

Proof. It suffices to show that $H^1(u^*T\mathbb{P}^k) = 0$. We apply the Euler short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{m+1} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0$$

Restricting this to a cruve C in the moduli space we get a long exact sequence in cohmology

$$\ldots \longrightarrow H^1(\mathcal{O}_{\mathbb{P}}) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^n}(1)^{m+1}|_C) \longrightarrow H^1(T_{\mathbb{P}^n}|_C) \longrightarrow 0$$

So if we can show that $H^1(\mathcal{O}_{\mathbb{P}^n}(1)^{m+1}|_C) = 0$ we're done. Note that $H^1(\mathcal{O}_{\mathbb{P}^n}(1)|_C) = \mathcal{O}_{\mathbb{P}^1}(d)$. Moreover, by Serre duality we know that

$$H^1(\mathcal{O}_{\mathbb{P}^1}(d)) \cong H^0(\mathcal{O}_{\mathbb{P}}(d)^* \otimes \omega_{\mathbb{P}^1})$$

Now, $\omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}}(-2)$ and $\mathcal{O}_{\mathbb{P}^1}(d)^* = \mathcal{O}_{\mathbb{P}^1}(-d)$ so

$$\mathcal{O}_{\mathbb{P}^1}(d)^* \otimes \omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-d-2)$$

Which admits no sections, so the cohomology in degree 0 vanishes, giving us

$$H^1(\mathcal{O}_{\mathbb{P}^1}(d)) \cong H^0(\mathcal{O}_{\mathbb{P}}(d)^* \otimes \omega_{\mathbb{P}^1}) = 0$$

□

2.1.3 Moduli of curves in a Hypersurface

Now consider a curve with a Kuranishi chart. Fix $X \subset \mathbb{P}^m$ of degree d, and assume that $X = s^{-1}(0)$ for $s \in H^0(\mathcal{O}_{\mathbb{P}^m}(\ell))$ and $m > 3$ so that by the Lefschetz Hyperplane theorem there is a well-defined $H_2(X)$-class of degree d. We consider

$$\overline{\mathcal{M}}_{0,n}(X, d) \hookrightarrow \overline{\mathcal{M}}_{0,n}(\mathbb{P}^n, d)$$

The goal is to define

$$[\overline{\mathcal{M}}_{0,n}(X, d)]_{\text{vir}} \in H_*(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^n, d))$$

Lemma 3

We have

$$\dim_{\mathbb{C}} \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d) = d(m+1) + (m-3) + n$$

As well as

$$\text{vdim}_{\mathbb{C}} \overline{\mathcal{M}}_{0,1}(X, d) = d(m-1+\ell) + (m-1) - 3 + n$$

Proof. By the virtual dimension formula we have that

$$\text{vdim}_{\mathbb{C}}\overline{\mathcal{M}}_{0,1}(X,d) = c_1(TX)A + (m-1-3)(1-0) + n$$

We just need to compute $c_1(TX)$. Using the short exact sequence

$$0 \longrightarrow TX \longrightarrow T\mathbb{P}^n|_x \longrightarrow \nu X \longrightarrow 0$$

We get that $c_1(\nu X) = \text{PD}([x]) = \ell[\mathbb{P}^{m-1}]$, so $c_1(TX) = c_1(\mathbb{P}^m) - c_1(\nu X)$ which is nothing but $\text{PD}((m+1)[\mathcal{P}^{m-1}] - \ell[\mathbb{P}^{m-1}])$ resulting in $d \cdot [\mathbb{P}^1]$. □

We want to identify the virtual fundamental cycle $[\overline{\mathcal{M}}_{0,n}(X,d)]_{\text{vir}} \in H_*(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^n, d))$ as the euler class of an obstruction bundle or equivalently the VFC of a Kuranishi chart. Take the bundle dual to X, $\mathcal{O}_{\mathbb{P}^m}(\ell) \to \mathbb{P}^m$. Given $u : \mathbb{P}^1 \to \mathbb{P}^m$ a J-holomorphic curve of degree d, we can take a pullback and get

$$u^*\mathcal{O}_{\mathbb{P}^m}(\ell) = \mathcal{O}_{\mathbb{P}^1}(d\ell)$$

Then sections $H^0(u^*\mathcal{O}_{\mathbb{P}^m}(\ell))$ are fibers of a vector bundle/sheaf living over $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$

$$\begin{array}{ccccc} \pi_*\text{ev}^*\mathcal{O}(\ell) & \longleftarrow & \text{ev}^*\mathcal{O}(\ell) & \longrightarrow & \mathcal{O}_{\mathbb{P}^m}(\ell) \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d) & \xleftarrow{\pi} & \overline{\mathscr{C}}_{0,n}(\mathbb{P}^m, d) & \xrightarrow{\text{ev}} & \mathbb{P}^m \end{array}$$

Where $\pi_*\text{ev}^*\mathcal{O}(\ell)$ is the obstruction bundle in the Kuranishi chart. To get the section in the Kuranishi chart, we know that $X = s^{-1}(0)$ for $s \in \Gamma(\mathcal{O}_{\mathbb{P}^n}(\ell))$. By applying push-pull to a section s we get a section σ of $\pi_*\text{ev}^*\mathcal{O}(\ell)$ with the property that

$$\sigma^{-1}(0) = \overline{\mathcal{M}}_{0,n}(X,d) \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)$$

Explicitly we have that $\sigma^{-1}(0) = u^*s$. So we can define the VFC of this moduli space as

$$[\overline{\mathcal{M}}_{0,n}(X,d)]_{\text{vir}} := e(\pi_*\text{ev}^*\mathcal{O}(\ell)) \cap [\overline{\mathcal{M}}_{0,n}(\mathbb{P}^m, d)]$$

2.2 Mohan Swaminathan - GKC for genus 0 curves

2.2.1 Motivation

The goal of the talk is to explain the following theorem

Theorem 8: AMS '21

Suppose (X,ω) is a closed sympletic manifold, J is an ω-tame ACS, $A \in H_2(X,\mathbb{Z})$, $n \geq 0$. Then $\overline{\mathcal{M}}_{0,n}(X, A; L)$ has a natural equivalence class of GKCs of the exp vritual dimension. If we change J to J', the the charts are cobordant.

2.2.2 Line Bundles on the Target

Given the symplectic form ω, we can approximate if by Ω which still tames J, but $[\Omega] \in H^2(X, \mathbb{Q})$. By clearing denominators, we can assume $[\Omega] \in H^2(X, \mathbb{Z})$. This gives us a smooth complex line bundle L_Ω over X with $c_1(L_\Omega) = [\Omega]$. We can then choose a metric and a connection on L_Ω such that the curvature form is $-2\pi i\Omega$. This is the first auxiliary choice in the construction

2.2.3 Framed Curves

Suppose $u : \Sigma \to X$ is a stable J-holomorphic genus 0 curve in the class of A. Let $d := [\Omega]A \geq 1$. Then we get u^*L_Ω and regard it as a holomorphic line bundle on Σ using $(u^*\nabla)^{0,1}$ as the $\overline{\partial}$-operator. First an observation:

- $\deg(u^*L_\Omega)$ is non-negative on each component of Σ, and positive on each unstable component of Σ

Now, choose a basis of $H^0(u^*L_\Omega)$ labeled $F = (f_0, ..., f_d)$. Now, there is a $\mathrm{GL}_{d+1}(\mathbb{C})$ worth of choice of F. Define the following matrix

$$\mathcal{H}(\Sigma, u, F) = \left(\int_\Sigma \langle f_i, f_j \rangle u^*\Omega \right)_{i,j}$$

Lemma 4

$\mathcal{H}$ as defined above is positive-definite

Definition 17: Framed Genus 0 Curve

A “framed genus 0 curve” in X is a triple (Σ, u, F) where

1. Σ is a nodal genus 0 curve
2. $u : \Sigma \to X$ is a smooth map in class A such that

$$\int u^*\Omega \geq 0$$

   On each component and strictly positive on unstable components
3. $F = (f_0, ..., f_d)$ is a choice of basis of $H^0(\Sigma, u^*L_\Omega)$ such that $\mathcal{H}(\Sigma, u, F)$ is positive definite.

We say that (Σ, u, F) is equivalent to (Σ', u', F') if there exists a biholomorphism $\varphi : \Sigma \to \Sigma'$ such that the diagram

$$\begin{array}{ccc} \Sigma & & \\ & \overset{u}{\searrow} & \\ \varphi \downarrow & & X \\ & \underset{u'}{\nearrow} & \\ \Sigma' & & \end{array}$$

commutes, and that $\varphi^* F' = F$. Observe that any framed curve (Σ, u, F) gives us a degree d, nondegenerate map

$$\phi_F : \Sigma \to \mathbb{P}^d$$

Where $\phi_F = [f_0 : ... : f_d]$. Now, let $\overline{\mathcal{M}}^*_{0,0}(\mathbb{P}, d) \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}, d)$ be the subset consisting of non-degenerate stable maps. This subset is a smooth variety of the expected dimension. Moreover there is a universal family sitting on top of this

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\text{ev}} & \mathbb{P}^d \\ {\scriptstyle\pi}\downarrow & & \\ \overline{\mathcal{M}}^*_{0,0}(\mathbb{P}^n, d) & & \end{array}$$

With $\mathscr{C}$ again being a smooth variety.

2.2.4 Achieving Transversality

Choose:

- Fix a relatively ample line bundle L on $\mathscr{C} \to \overline{\mathcal{M}}^*_{0,0}(\mathbb{P}^d, d)$ such that the $U(d+1)$ action on $\mathscr{C}$ lifts to L, and a metric on L that is invariant under $U(d+1)$.
- A complex linear connection on both $T^{*(0,1)}\mathscr{C} = \overline{T^*\mathscr{C}}$ invariant under $U(d+1)$ and on TX
- An integer $k \gg 0$

Definition 18: GKC for genus 0 curve

Let $\mathcal{K} = (G, T, \mathcal{E}, s, \psi)$ where

1. The thickening T is the space of tuples (Σ, u, F, η) where (Σ, u, F) is a framed curve, and $\eta \in E_k(\Sigma, u, F)$ where

$$E_k(\Sigma, u, F) := H^0(\Sigma, u^*TX \otimes \iota_F^*(T^{*(0,1)}\mathscr{C} \otimes L^{\otimes k})) \otimes_{\mathbb{C}} \overline{H^0(\Sigma, \iota_F^* L^{\otimes k})}$$

   such that

$$\overline{\partial}_J u + \langle \eta \rangle \circ d\widetilde{\iota_F} = 0$$

2. The obstruction bundle $\mathcal{E} \to T$ has fiber over (Σ, u, F, η):

$$E_k(\Sigma, u, F) \oplus \mathcal{H}_{d+1}$$

   Where $\mathcal{H}_{d+1}$ is the space of $(d+1) \times (d+1)$ hermitian matrices.

3. The obstruction section $s : T \to \mathcal{E}$ such that

$$s(\Sigma, u, F, \eta) = (\eta, \log \mathcal{H}(\Sigma, u, F))$$

4. $\psi : S^{-1}(0)/G \to \overline{\mathcal{M}}_{0,0}(X, A; J)$ is the map forgetting the framing.

Now, given $(\Sigma, u, F, 0) \in s^{-1}(0)$ and fixing Σ, F we can consider

$$\Omega^0(\Sigma, u^*TX) \oplus E_k(\Sigma, u, F) \xrightarrow{Du \oplus \langle\rangle \circ d\tilde{\iota}_F} \Omega^{0,1}(\tilde{\Sigma}, u^*TX)$$

We claim that this map surjects.

2.3 Amanda Hirschi - Kontsevich-Manin Axioms

2.3.1 Introduction

Consider first moduli spaces of stable curves. There are natural maps:

1. $\pi_k : \overline{\mathcal{M}}_{0,k} \to \overline{\mathcal{M}}_{0,k-1}$ the map that forgets a marked point
2. $S_n \times \overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,n}$ permuting marked points
3. $\varphi : \overline{\mathcal{M}}_{0,??} \times \overline{\mathcal{M}}_{??,??} \to \overline{\mathcal{M}}_{??,??}$ called the clutching map

Lemma 5

For each partition $S = S_? \sqcup S_?$ with $|S_i| \geq 2$ there exists a clutching map φ_s. Its image is a divisor $V_?$

Lemma 6

The Poincaré duals $\{\gamma_? := \mathrm{PD}(V_?)\}_S$ generate $H^*(\overline{\mathcal{M}}_{??}, \mathbb{R})$ as a ring

Example 2.1. When $n = 4$, we have that $S^2 \cong \overline{\mathcal{M}}_{0,4}$ thus $\gamma_S = \gamma_{S'}$ for any S, S'

By the previous talk, we can associate to $\overline{\mathcal{M}}^?_{??}(X, A)$ a global Kuranishi chart

$$K = (G, T, \mathcal{E}, s)$$

Crucially, there exists a submersion

$$T \to B(?) \subset \overline{\mathcal{M}}_{0,0}(????, d)$$

For some $d \geq 0$. Now, let

$$\pi_x : \overline{\mathcal{M}}_{0,n}(X, A; J) \to \overline{\mathcal{M}}_{0,0}(X, A; J)$$

Be the forgetful map. Define

$$B_?(d) := \pi^{-1}(B(d)) \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^d, d)$$

Lemma 7

The tuple

$$K_? := G, B_?(d) \times T, B_?(d) \times \mathcal{E}, \mathrm{id} \times s)$$

Is a GKC for $\overline{\mathcal{M}}_{0,?}(X, A; J)$

The upshot of this is that we have a virtual fundamental class $[\overline{\mathcal{M}}^?_{0,n}(X, A)]^{\mathrm{vir}}$ for any n. We have an evaluation map

$$\mathrm{ev} : \overline{\mathcal{M}}^J_{0,n}(X, A) \to X^?$$

and a stabilization map

$$\mathrm{st} : \overline{\mathcal{M}}^J_{0,n}(X, A) \to \overline{\mathcal{M}}_{0,?}$$

If $? \leq 2$, we take $\overline{\mathcal{M}}_{0,?}$ formally to be a point

Definition 19

The "Gromov-Witten" classes of (X, ω) on the homology classes

$$GW^{X,\omega}_{???} := (\mathrm{ev} \times \mathrm{st})???????????$$

I MISSED A BUNCH IN THIS SLIDE

2.3.2 The Easy Axioms

1. Effective: If $\omega(A) < 0$, then $GW^?_? = 0$
2. Grading: $GW^?_?$ has degree $d = \dim_{\mathbb{R}}(X) + 2c_1(A) + 2(n-3)$
3. Symmetry: Observe that S_n acts on $\overline{\mathcal{M}}_{?,?}$ by permuting the marked points. Then we have for any $\rho \in S_n$ that

$$\langle \alpha_{\rho(1)}, ..., \alpha_{\rho(n)} \rangle^?_? = (-1)^? \langle \alpha_1, ..., \alpha_n \rangle^?_?$$

   Which tells us VFC is S_n-invariant

4. ?????: If $A = 0$, then

$$\langle \alpha_1, ..., \alpha_n \rangle^?_? = \begin{cases} b \cdot \langle \alpha_1 \smile ... \smile \alpha_n, [X] \rangle & \sigma = b[\text{pt}] \\ 0 & \text{else} \end{cases}$$

   Thus $\overline{\mathcal{M}}_{0,n} = X \times \overline{\mathcal{M}}_{0,n}$ is regular, so VFC=VC.

2.3.3 Fundamental Class Axiom

Let $\pi_n : \overline{\mathcal{M}}_{0,n}(X, A; J) \to \overline{\mathcal{M}}_{0,?}(X, A; J)$ just forget the nth marked point. Then if $n \geq ?$, then

$$\langle \alpha_1, ..., \alpha_?, ??? \rangle^{X,\omega}_{??} = \langle \alpha_1, ..., \alpha_{m-1}, ; (\pi_k)_* \sigma \rangle^{X,\omega}_?$$

The intuition here is that there's no constraint on the nth marked points. In terms of VFCs, this is equivalent to

$$(\pi_n)_*(\text{st}^*\text{PD}(\sigma) \cap [\overline{\mathcal{M}}_{0,n}(X, A; J)]^{\text{vir}}) = \text{st}^*\text{PD}((\pi_n)_*\sigma) \cap [\overline{\mathcal{M}}_{0,n-1}(X, A; J)]^{\text{vir}}$$

if π_n and st were submersions, this would follows from general algebraic topology. Given a GKC, the proof is essentially the same.

2.3.4 The Splitting Axiom

Write $\text{PD}(\Delta_x) = \sum_i \beta_i \times \gamma_i$ and let

$$\varphi : \overline{\mathcal{M}}_{0,?} \times \overline{\mathcal{M}}_{0,?} \to \overline{\mathcal{M}}_{0,?}$$

be a clutching map. Then

$$\langle \alpha_1, ..., \alpha_n; \varphi_?(\sigma_0 \times \sigma_n) \rangle^{X,\omega}_{???} = \sum_{???} \sum_i \langle \alpha_?, ..., \alpha_?; \beta_i, \sigma_? \rangle^{X,\omega}_? \langle \gamma_i, \alpha_?, ..., \alpha_n, \sigma? \rangle^{X,\omega}_{????}$$

The intuition for this is to suppose that $\sigma_i = [\overline{\mathcal{M}}_{0,?????}]^{\text{vir}}$, and then there's some picture??????. Here's a sketch of the proof: The map φ usually does not lift to a map

$$K_? \times K_? \to K_?$$

Instead, make a new GKC for $\overline{\mathcal{M}}(A_?, A_?) := \overline{\mathcal{M}}_{????}(X, A_?; J) \times \overline{\mathcal{M}}_?(X, A_?; J)$. Then set

$$B_{????}(d_?, d_?) := \overline{\mathcal{M}}_?(\mathbb{P}^d, d_0) \times \overline{\mathcal{M}}_{???}(\mathbb{P}^d, d_1) \cap \varphi^{-1}(B_n(d))$$

Which is regular, then

$$\tau_{??????} := B_?(d_?, d_?) \times_{B_n(d)} \tau$$

is a manifold with $\tilde{\varphi} : \tau_{???} \to \tau_n$.

Lemma 8

$K_{????} := (G, \tau_{??}, \tilde{\varphi}^*\mathcal{E}_k, \tilde{\varphi}^* s_n)$ is a GKC for $\overline{\mathcal{M}}(A_?, A_?)$ equivalent to **SOMETHING I DONT KNOW WHAT * IS**

2.3.5 Divisor Axiom

Suppose that $n \geq ?$ and $|\alpha_?| = 2$. Then

$$\langle \alpha_1, ..., \alpha_n; [\overline{\mathcal{M}}_{0,n}] \rangle^{X,\omega}_{???} = \langle \alpha_?, A \rangle \cdot \langle \alpha_?, ..., \alpha_{n-1}; [\overline{\mathcal{M}}_{????}] \rangle^{X,\omega}_{???}$$

The intuition is that is $\alpha_? = \text{PD}(Y)$, for $Y \subset X$ a divisor, a generic curve $u : S^2 \to X$ has $Y \cdot u = \langle \alpha_?, A \rangle$, thus there exists $\langle \alpha_?, A \rangle$ many places for the nth marked points. Here's a proof sketch:

1. We can arrange for $\text{ev}_\alpha : \tau_? \to X$ to be a submersion
2. $\tau_Y := \text{ev}_?^?(Y)$ is a manifold with $\dim(\tau_Y) = \dim(\tau_{??})$
3. The VFC of $(G, \tau_Y, \mathcal{E}_?|_{\tau_Y}, s_n|_{\tau_Y})$ is $\text{ev}_?^*$**I MISSED A BUNCH IN THIS SLIDE TOO**

2.4 Mark McLean - GKS: Some Problems to Solve

2.4.1 Problems

Here are some problems:

1. Try to find simpler ways to build GKCs for moduli spaces of curves. These were invented to find something "natural" to surject onto the cokernel of the $\overline{\partial}$ operator. Also, we wanted to identify the domain of a curve with an element of some explicit family of nodal curves.

2. Constructing nice GKCs for Floer Theory is still difficult. When you want to prove a relation, like $\partial^2 = 0$, you need to express the boundary of a moduli space as a fiber product of GKCs via a morphism. Also when you forget marked points you want a morphism between the charts such that
   (a) The order in which you forget does not matter
   (b) evaluation maps are still submersions
   (c) It is still compatible with the fiber product maps above
3. If out symplectic manifold has special properties, do the charts inherit something nice too? If i have a "nice" space X, do the GKCs for Gromov-Witten Theory look "nice"? Does this put constraints on GW invariants? For instance does positive Ricci curvature put constraints in the GW invariants? Looking at a primitive classes would be a start since one does not have to wrestle with gluing and smooth structures
4. Can we construct GKCs for other Floer theories, like Instanton Floer?

Doing all of these in a nice way is tricky, but we'll focus on the second. Let's look at the Floer Theory problem. Recall that a GKC is a tuple (G, T, E, s) where G is a compact Lie Group, $E \to T$ is a G-vector bundle with G equivariant section s, and G acts faithfully and semi-freely on T.

Definition 20: Morphisms of GKCs

A morphism of GKCs $(G, T, E, s) \to (G', T', E', s')$ consists of an equivalence class of commutative diagrams

$$\begin{array}{ccccc} E & \longleftarrow & \pi^*E & \longrightarrow & E' \\ {\scriptstyle s}\uparrow\downarrow & & {\scriptstyle \pi^*s}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle s'} \\ T & \longleftarrow & P & \xrightarrow[f]{} & T' \end{array}$$

Where π is a G-equivariant principal G'-bundle, f is a G'-equivariant map

For germ equivalence we have

$$\begin{array}{ccccc} E'|_U & \xleftarrow{\text{id}} & E'|_U & \longrightarrow & E' \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle p} & & {\scriptstyle s}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle s'} \\ T'|_U & \xleftarrow[\text{id}]{} & T'|_U & \longrightarrow & T' \end{array}$$

For stabilization we have

$$\begin{array}{ccccc} E & \xleftarrow{\text{id}} & E & \longrightarrow & p'^*E \oplus E' \\ {\scriptstyle s}\uparrow\downarrow{\scriptstyle p} & & {\scriptstyle s}\uparrow\downarrow & & {\scriptstyle p'^*p}\downarrow\uparrow{\scriptstyle p^*s+\Delta} \\ T & \xleftarrow[\text{id}]{} & T & \xrightarrow[0-\text{section}]{} & E' \end{array}$$

We want to construct a "nice" system of GKCs for Floer Theories, with morphisms between them corresponding to gluing maps as well as forgetful maps. However, for simplicity let us just look at compatible systems for genus g curves. It would be nice for us to have:

1. If we fix $g, h \in \mathbb{N}$ and each $\beta \in H_2(X, \mathbb{Z})$ we'd like to assign a GKC $T_{h,g,\beta}$ to the moduli space of genus zero curves with h marked points representing β

2. These have evaluation maps $\text{ev} : T_{h,g,\beta} \to X^h$ which are submersions

3. For each bijection $\phi : \{1, ...h_1\} \times \{1, ..., h_2\} \to \{1, ..., h_1 + h_2\}$, we have a corresponding map of $GKCs$:

$$G_\phi : T_{h_1+1,g_1,\beta_1} \times T_{h_2+1,g_2,\beta_2} \to T_{h_1+h_2,g_1+g_2,\beta_1+\beta_2}$$

   For each $i \in \{1, ..., h\}$, we have a forgetful morphism $F_i : T_{h,g\beta} \to T_{h-1,g,\beta}$ forgetting the i-th marked point.

2.5 Mohan Swaminathan - GKC for Higher Genus Holomorphic Curves

2.5.1 New Issues in Higher Genus

Suppose we started with a map $u : \Sigma \to \mathbb{P}^n$. If we knew the degree of the map, pulling back $\mathcal{O}(1)$ we know exactly the line bundle we get in the genus zero case. In higher genus this goes wrong: For a nodal curve Σ of genus $g > 0$, there exists holomorphically nontrivial but topologically trivial line bundles. This also has to do with the fact that $H^1(\Sigma, \mathcal{O}_\Sigma) \neq 0$. This causes a number of issues. Suppose $u : \Sigma \to X$ is a stable J-holomorphic curve of genus $g \neq 0$

1. $\dim H^0(\Sigma, u^*|_\Omega)$ can jump as u varies! For example if we have a torus attached to a sphere, and the torus gets mapped to a point, i.e. it's a ghost component. The remedy for this is to replace L_Ω by another natural choice without this issue. What we do is consider $\omega_\Sigma \otimes (u^* L_\Omega)^{\otimes 3}$. The 3 is enough to stay positive if we need to subtract 2 as in Julian's talk.

2. Suppose we "frame" $u : \Sigma \to X$ using a basis F of $H^0(\Sigma, \omega_\Sigma \otimes (u^* L_\Omega)^{\otimes 3})$. This gives us $\phi_F : \Sigma \to \mathbb{P}^N$, giving us an isomorphism

$$\phi_F^* \mathcal{O}(1) \cong \omega_\Sigma \otimes (u^* L_\Omega)^{\otimes 3}$$

   Now, considering $[\phi_F] \in \overline{\mathcal{M}}_g^*(\mathbb{P}^n, d)$ **What happened here?**

2.5.2 Picard Groups of (families of) Curves

Definition 21: Picard Group

Let Σ be a nodal curve, and deifine

$$\text{Pic}(\Sigma) = \{L \to \Sigma \text{ holomorphic line bundles}\}/\sim$$

Which is a group under tensoring, and $\text{Pic}^0(\Sigma)$ is the subgroup of topologically trivial line bundles. Suppose $\pi : \mathscr{C} \to S$ is a family of nodal curves, then

$$\text{Pic}(\mathscr{C}/S) = \{(s, [L_s] : s \in S, [L_s] \in \text{Pic}(\mathscr{C})$$

And $\text{Pic}^0(\mathscr{C}/S)$ is fiberwise Pic^0 from before

Now, suppose we have a single curve Σ, and consider the short exact sequence of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_\Sigma \xrightarrow{\exp} \mathcal{O}^*_\Sigma \longrightarrow 0$$

The first part of the long exact sequence in cohomolgoy is

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^* \longrightarrow 0$$

After this zero we have

$$0 \longrightarrow H^1(\Sigma, \mathbb{Z}) \longrightarrow H^1(\Sigma, \mathcal{O}_\Sigma) \longrightarrow H^1(\Sigma, \mathcal{O}^*_\Sigma) \xrightarrow{c_1} H^2(\Sigma, \mathbb{Z}) \longrightarrow 0$$

So $\text{Pic}^0(\Sigma) \cong \frac{H^1(\Sigma, \mathcal{O}_\Sigma)}{H^1(\Sigma, \mathbb{Z})}$ where the isomorphism is induced by the exponential map. Now, for a family of curves $\pi : C \to S$, then $s \mapsto H^1(C_s, \mathcal{O}_{C_s})$ defines a rank g holomorphic vector bundle on S, which we'll denote $\mathbb{H}^*_{C/S}$. We then get

$$\mathbb{H}^*_{C/S} \to \text{Pic}^0(C/S)$$

Is an analytic isomorphism in a neighborhood of the zero section

2.5.3 Construction

We want to construct a GKC for $\overline{\mathcal{M}}_{g,0}(X, A; J)$, so we have to make some choices:

1. Choose L_Ω as before and define $d = [\Omega] \cdot A \geq 0$
2. For every $u : \Sigma \to X$ we define $L_u = \omega_\Sigma \otimes u^* L_\Omega^{\otimes 3}$
3. Choose an integer $p \gg 1$, and define $m = p(2g - 2 + 3d) = \deg(L_u^{\otimes p})$, $N = m - g = h^0(L_u^{\otimes p}) - 1$, $\mathscr{G} = \text{PGL}_{N+1}(\mathbb{C})$ and $G = \text{PU}(N + 1)$.

Now we get a new notion of framed curve: A tuple

$$(\Sigma, u, L, F)$$

Where

1. $u : \Sigma \to X$ is a stable C^∞ map of genus g
2. L is a holomorphic line bundle of multi-degree 0
3. $F = (f_0, ..., f_n)$ is a $\mathbb{C}$-basis of $H^0(\Sigma, L_u^{\otimes p} \otimes L)$

In addition we need

1. A $\mathbb{C}$-linear connection ∇^X on TX
2. A large integer $k \gg 1$
3. $\lambda : \{\text{Framed stable map}\} \to \mathscr{G}/G$ a $\mathscr{G}$-equivariant map

Definition 22: GKC

$K = (G, T, \mathcal{E}, s, \psi)$ where

1. T is the module space of tuples

$$(\Sigma, u, L, F, \eta, \alpha)$$

Which consists of framed curves with some extra data: $\eta \in E_k(\Sigma, u, L, F)$ and $\alpha \in H^1(\Sigma, \mathcal{O}_\Sigma)$ such that

$$\bar{\partial} u + \langle \eta \rangle \circ d\tilde{\iota}_F = 0$$

and $L = \exp(\alpha)$

2. $\mathcal{E} \to T$ has fiber over $(\Sigma, u, L, F, \eta, \alpha)$ given by

$$H^1(\Sigma, \mathcal{O}_\Sigma) \oplus E_k(\Sigma, u, L, F) \oplus \mathfrak{su}(N+1)$$

3. $s : T \to \mathcal{E}$ is defined as

$$s(\Sigma, u, L, F, \eta, \alpha) = (\alpha, \eta, i \log \lambda(...))$$

4. $G = \mathrm{PU}(N+1)$