Linear Systems and Peak Sections

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In this talk, we will introduce some algebro-geometric notions on linear systems on curves, and their roles in proving transversality for moduli spaces. We will roughly follow [Har77].

1. Linear Systems

Let X be a nonsingular projective variety over a ring k and let $\mathcal{L}$ be a line bundle on X. For each non-zero global section $s \in \Gamma(X, \mathcal{L})$, we can define the **divisor of zeroes** of s, denoted $D = (s)_0$, to be the hypersurface defined by the equation $s = 0$. Geometrically, we have

1.1 Proposition. Let D_0 be a divisor on X with $\mathcal{L} = \mathcal{L}(D_0)$ be the corresponding line bundle on X, then

(a) for each nonzero $s \in \Gamma(X, \mathcal{L})$, the divisor of zeros $(s)_0$ is an effective divisor linearly equivalent to D_0;

(b) every effective divisor linearly equivalent to D_0 is $(s)_0$ for some $s \in \Gamma(X, \mathcal{L})$;

(c) two sections $s, s' \in \Gamma(X, \mathcal{L})$ have the same divisor of zeros if and only if there is a $\lambda \in k^*$ with $s' = \lambda s$.

Based on the definition, we have

1.2 Definition. A **linear system** $\mathfrak{d}$ on X is the projectivization of a linear subspace of $\Gamma(X, \mathcal{L})$. The **dimension** of $\mathfrak{d}$ is the dimension of the corresponding projective space.

A point $p \in X$ is a **base point** of a linear system $\mathfrak{d}$ if $p \in \operatorname{supp} D$ for all $D \in \mathfrak{d}$.

It's clear from the definition that

1.3 Lemma. p is a base point of $\mathfrak{d}$ if and only if for all $s \in \mathfrak{d}$, $s(p) = 0$. In particular, $\mathfrak{d}$ is base-point free if and only if $\mathcal{L}$ is generated by global sections.

Consider the largest linear system $\mathfrak{d} = \mathbb{P}\Gamma(X, \mathcal{L})$. A choice of basis of $\mathfrak{d}$ induces a map $\varphi = (s_1 : \cdots : s_n) \colon X \to \mathbb{P}^{n-1}$ if $\mathfrak{d}$ is base-point free. Further more,

1.4 Definition. $\mathfrak{d}$ is said to **separate points** if for any two distinct points $p, q \in X$, there is $D \in \mathfrak{d}$ such that $p \in \operatorname{supp} D$ and $q \notin \operatorname{supp} D$.

$\mathfrak{d}$ is said to **separate tangent vectors** if given a closed point $p \in X$ and a tangent vector $t \in T_pX$, there is $D \in \mathfrak{d}$ such that $p \in \operatorname{supp} D$ but $t \notin T_p(D)$.

1.5 Proposition. φ is a closed immersion if and only if $\mathfrak{d}$ is base-point free, separates points and tangent vectors.

1.6 Definition. A line bundle $\mathcal{L}$ over X is **ample** if there exists m so that $\mathcal{L}^{\otimes m}$ is very ample, i.e. it's base-point free and the corresponding morphism φ is a closed immersion.

1.7 Proposition. Equivalently, $\mathcal{L}$ is ample if and only if for every coherent sheaf $\mathcal{F}$ on X, there is an m_0 so that for all $m \geq m_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is generated by global sections.

Recall that there is a sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$ dual to the tautological line bundle $\mathcal{O}(-1)$, which under immersion φ pulls back to a line bundle on X.

1.8 Proposition. We have $\mathcal{L} \simeq \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$ if $\mathcal{L}$ is very ample.

We conclude this section with Serre's vanishing theorem:

1.9 Theorem (Serre). Let X be a projective scheme over a noetherian ring A, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over $\operatorname{Spec} A$.

(a) for each $i \geq 0$, $H^i(X, \mathcal{F})$ is a finitely generated A-module;

(b) there is an integer n_0, depending on $\mathcal{F}$, such that for each $i > 0$ and each $n \geq n_0$, $H^i(X, \mathcal{F}(n)) = 0$.

2. Divisors on Curves

Now we turn to divisors on algebraic curves. Given a nonsingular projective curve X and a line bundle $\mathcal{L}$ on X, the corresponding divisor of zeroes of global sections of $\mathcal{L}$ are formal linear combinations of closed points on X. Given $p \in X$, the map $D \mapsto D + p$ of divisors induces a map of linear systems $|D| \to |D + p|$

2.1 Proposition. Let D be a divisor on X. Then

(a) the complete linear system $|D|$ has no base points if and only if for every point $p \in X$,

$$\dim |D - p| = \dim |D| - 1;$$

(b) D is very ample if and only if for every two points $p, q \in X$ (including the case $p = q$),

$$\dim |D - p - q| = \dim |D| - 2.$$

Proof. We have the following short exact sequence of line bundles

$$0 \to \mathcal{L}(D - p) \to \mathcal{L}(D) \to k(p) \to 0$$

which induces an exact sequence of global sections

$$0 \to \Gamma(X, \mathcal{L}(D - p)) \to \Gamma(X, \mathcal{L}(D)) \to k,$$

so that $\dim |D - p|$ is either $\dim |D|$ or $\dim |D| - 1$ and the map $\varphi\colon |D - p| \to |D|$ is injective. φ is surjective if and only if p is a base point, and we concludes (a).

If D is very ample, D has no base points, so we must have $\dim |D - p - q| = \dim |D| - 2$. Conversely, it follows from the condition and (a) that $|D|$ has no base points, and we need to show it separates points and tangent directions. Let $p, q \in X$ be distinct points, the existence of sections on $|D|$ non-vanishing on p is equivalent to saying p is not a base point of $|D|$, so equivalently $\dim |D - p| = \dim |D| - 1$. Therefore separating points is equivalent to p, q not being basepoints of $|D|$, which is clear from the condition. Now fix $p \in X$, note that $\dim T_p X = 1$, tangent vector separation is equivalent to saying the existence of a divisor $D' \in |D|$ such that p has multiplicity 1, since $\dim T_p D' = 0$ if p has multiplicity 1 in D' and 1 if having higher multiplicity. But having multiplicity 1 is equivalent to saying p is not a basepoint of the linear system $|D - p|$, which is equivalent to $\dim |D - 2p| = \dim |D| - 2$. Therefore D is very ample and we have shown (b). □

We conclude this section with a result on reducing the dimension of the projective space that X can be closedly immersed into. Assume that we already have a closed immersion $X \hookrightarrow \mathbb{P}^n$, and let $O \in \mathbb{P}^n \setminus X$ be any point. We define a map $\varphi\colon X \to \mathbb{P}^{n-1}$ as follows: put $O = [1 : 0 : \cdots : 0] \in \mathbb{P}^n$ and $D_\infty = \{[0 : a_1 : \cdots : a_n] | a_i \in k\}$ be the divisor at infinity. For each $x \in X$, we let $\varphi(x)$ be the intersection of the unique line passing through O and x with D_∞. This map is clearly well-defined and is algebraic.

2.2 Definition. We define the **secant line** determined by $p, q \in X$ to be the line in $\mathbb{P}^n$ joining p and q. If p is a point of X, the **tangent line** to X at p is the unique line $L \subseteq \mathbb{P}^n$ passing through p, whose tangent line $T_p(L)$ is equal to $T_p(X)$ as a subspace of $T_p(\mathbb{P}^n)$.

2.3 Proposition. φ is a closed immersion if and only if

(a) O is not on any secant line of X, and

(b) O is not on any tangent line of X.

Proof. Consider the dimension $n - 1$ linear system in $|\mathcal{O}_{\mathbb{P}^n}(1)|$ with O as the common base point. Intersection of any divisor $D \in |\mathcal{O}_{\mathbb{P}^n}(1)|$ with X gives a divisor on X, with φ the corresponding morphism into $\mathbb{P}^{n-1}$. The condition listed in the Proposition is then equivalent to the separation of points and tangent vectors condition, hence the conclusion. □

3. Framed Curves

In this section, we will give a coarse analysis of moduli space of pseudo-holomorphic spheres in $\mathbb{P}^d$ of degree d. Write

$$\mathcal{M}_{0,0}(\mathbb{P}^d, d)$$

for the moduli space. Let $\mathcal{F}$ be the Zariski open subset of $\mathcal{M}_{0,0}\left(\mathbb{P}^d, d\right)$ consisting of nodal holomorphic spheres not contained in any hyperplanes of $\mathbb{P}^d$, and let

$$\mathrm{univ}\colon \mathcal{C} \to \mathcal{F}$$

be the **universal curve**, where the preimage $\mathrm{univ}^{-1}(F)$ is the curve F in $\mathbb{P}^d$.

3.1 Lemma. Both $\mathcal{C}$ and $\mathcal{F}$ are quasi-projective smooth varieties.

Proof. Note that $\mathcal{M}_{0,0}\left(\mathbb{P}^d, d\right)$ is a projective orbifold, where the smooth locus consists of points of trivial automorphism group. Given an element $u\colon \Sigma \to \mathbb{P}^d$ of $\mathcal{F}$, the pull-back $u^*\mathcal{O}_{\mathbb{P}^d}(1)$ has degree d on Σ, and since $u(\Sigma)$ does not lie inside any hyperplanes, we get a canonical basis of sections $\{s_1, \cdots, s_d\} \subseteq H^0\left(\Sigma, u^*\mathcal{O}_{\mathbb{P}^d}(1)\right)$. Therefore we obtain a pair $(\Sigma, u^*\mathcal{O}_{\mathbb{P}^d}(1), \phi_F)$ where F comes from linear hyperplanes. Conversely, given such a pair (Σ, L, F) where L is a line bundle over Σ of degree d and F a frame of $H^0(\Sigma, L)$, a choice of basis of $H^0(\Sigma, L)$ induces an embedding of Σ into $\mathbb{P}^d$ not contained in any hyperplanes, so we have a bijection between $\mathcal{F}$ and the space of pairs (Σ, L, F).

The automorphism group of the pair (Σ, L, F) is an automorphism group of Σ that lifts to an automorphism of L and fixes the basis F, so should act on $H^0(\Sigma, L)$ trivially. Over each unstable component of Σ, L has strictly positive degree so that there are sections of L vanishing on all other components, and hence the automorphism group should fix the dual graph of Σ, and hence fixing each components. Since the dual graph is a tree, leaves are unstable and the automorphism group must fix the unstable component pointwise. This implies that the automorphism group of (Σ, L, F) is trivial, and the smoothness follows. □

3.2 Definition. Let Σ be a genus 0 nodal curve. A **domain map** is an inclusion map $\iota\colon \Sigma \hookrightarrow \mathcal{C}$ so that ι is an isomorphism onto a fiber of the universal curve over $\mathcal{F}$.

3.3 Lemma. Let Σ be a genus 0 nodal curve and let $\Omega \in \Omega^2(\Sigma)$ a closed 2-form representing an integral cohomology class. Then there exists a unique Hermitian line bundle on Σ up to isomorphism whose curvature is $-2\pi i\Omega$.

3.4 Definition. For any nodal connected genus zero curve Σ and for any 2-form Ω on Σ admitting an integral lift, we define L_Ω to be a Hermitian line bundle whose curvature is $-2\pi i\Omega$. We write $\langle -, - \rangle_\Omega$ for its associated Hermitian metric.

3.5 Definition. A **framed genus** 0 **curve** in X is a tuple (u, Σ, F) where

(a) Σ is a genus zero nodal curve;

(b) $u\colon \Sigma \to X$ is a smooth map representing β so that the degree of $L_{u^*\Omega}$ is strictly positive on each unstable component of Σ;

(c) $F = (f_0, \cdots, f_d)$ is a basis of $H^0(L_{u^*\Sigma})$ so that the Hermitian matrix

$$\mathcal{H}(u, \Sigma, F) := \left(\int_\Sigma \langle f_i, f_j \rangle_{u^*\Omega} \right)_{i,j=0,\cdots,d}$$

has positive eigenvalues.

4. Hörmander's Peak Sections and Transversality

In this section, we briefly explains the construction of "peak sections" that will play a role in achieving transversality for thickenings of moduli spaces of holomorphic curves. We explain the setting in a slightly more general way: consider a compact Kähler manifold $(Y, J_Y, \hat{\omega})$ without boundary of dimension m, and choose a Hermitian vector bundle $\hat{E}$ together with an ample line bundle L on Y. We write $\mathrm{Herm}(\hat{E}, \hat{E})$ for the bundle of hermitian endomorphisms of $\hat{E}$. With the given metric, cup product by $\hat{\omega}$ induces a dual map

$$\Lambda_{\hat{\omega}}\colon \bigwedge^{p,q} T^*Y \otimes \mathrm{Herm}(\hat{E}, \hat{E}) \to \bigwedge^{p-1,q-1} T^*Y \otimes \mathrm{Herm}(\hat{E}, \hat{E})$$

analoguous to the Lefschetz operator. On the other hand, recall that a hermitian connection ∇ is a map

$$\nabla\colon \hat{E} \to T^*Y \otimes \hat{E}$$

which preserves the hermitian metric on $\hat{E}$ and is holomorphic for J_Y. The curvature of ∇ can then be written as

$$R_{\hat{E}} = \nabla_{\hat{E}} \circ \nabla_{\hat{E}}\colon \hat{E} \to \wedge^2 T^*Y \otimes \hat{E},$$

which is equivalently a section of the bundle $\bigwedge^{1,1} T^*Y \otimes \overline{\mathrm{Herm}(\hat{E}, \hat{E})}$, so that $iR_{\hat{E}}$ is a section of $\bigwedge^{1,1} T^*Y \otimes \mathrm{Herm}(\hat{E}, \hat{E})$. Wedge sum by $iR_{\hat{E}}$ induces a map of bundles $\bigwedge^{p,q} T^*Y \otimes \mathrm{Herm}(\hat{E}, \hat{E}) \to \bigwedge^{p+1,q+1} T^*Y \otimes \mathrm{Herm}(\hat{E}, \hat{E})$.

4.1 Theorem (Hörmander). Let $p, q \in \mathbb{N}$ and suppose that the commutator

$$A := [iR_{\hat{E}}, \Lambda_{\hat{\omega}}]\colon \bigwedge^{p,q} T^*Y \otimes \mathrm{Herm}(\hat{E}, \hat{E}) \to \bigwedge^{p,q} T^*Y \otimes \mathrm{Herm}(\hat{E}, \hat{E})$$

is positive definite on each fiber. Let $c_{\hat{E}}$ be the L^∞ norm of A and define $c := c_{\hat{E}}^{-1}$. Then for each $g \in L^2\left(\wedge^{p,q} T^*Y \otimes \hat{E}\right)$ satisfying $\bar{\partial} g = 0$, there exists $f \in L^2\left(\wedge^{p,q-1} T^*Y \otimes \hat{E}\right)$ satisfying $\|f\|_{L^2} \leq c\|g\|_{L^2}$ and $\bar{\partial} f = g$.

Fix an effective divisor D, we can extend Hörmander's result to vector bundles of the form $\hat{E} \otimes \mathcal{O}(D)$ where we choose a hermitian metric on $\mathcal{O}_Y(D)$ making the commutator $[iR_{\hat{E}\otimes\mathcal{O}_Y(D)}, \Lambda_{\hat{\omega}}]$ positive definite on (p,q)-forms.

4.2 Lemma. If $g = 0$ in a small neighbourhood of D, tand we set $c = c^{-1}_{\hat{E}\otimes\mathcal{O}_Y(D)}$, then we can find f with $f|_D = 0$.

Proof. Note that if $g = 0$ in a small neighbourhood of D, then g belongs to sections of the line bundle $\hat{E} \otimes \mathcal{O}_Y(D)$, and Hörmander's result applies to give $f \in L^2\left(\wedge^{p,q-1}T^*Y \otimes \hat{E} \otimes \mathcal{O}_Y(D)\right)$ with $\bar{\partial}f = g$ and $\|f\|_{L^2} \leq c\|g\|_{L^2}$. Clearly $f|_D = 0$. □

The key Lemma of this section is the following "existence of peak sections":

4.3 Lemma. Fix an effective divisor $D \subseteq Y$ and a point $x \in Y \setminus D$. Let $e \in \hat{E}|_x$, and let δ_e be the Dirac delta section at x with value e. There are holomorphic sections s_k and $\check{s}_k$ of $\hat{E} \otimes L^k$ and L^k respectively, for $k \in \mathbb{N}$, so that

- $\langle s_k, \check{s}_k \rangle \to \delta_e$ in the sense of distributions as $k \to \infty$;
- s_k and $\check{s}_k$ vanish along D.

Proof. Based on [Tia90], there exists a sequence of holomorphic sections $(\check{s}_k)_{k\in\mathbb{N}}$ of L^k where the norm of $\check{s}_k$ converges to the Dirac delta function at x. Consider a smooth section $\sigma \in C^\infty(\hat{E})$ with $\sigma(x) = e$ and is holomorphic in a neighbourhood of x. Define $s'_k := \sigma \otimes \check{s}_k$, then we can find section $(g_k)_{k\in\mathbb{N}}$ of $(\hat{E} \otimes L^k)_{k\in\mathbb{N}} = \left(\bigwedge^{m,0} Y^*Y \otimes \hat{E}'_k\right)_{k\in\mathbb{N}}$ such that the L^2-norm of g_k goes to 0 as $k \to \infty$ and $\bar{\partial}g_k = \bar{\partial}s'_k$ in $\bigwedge^{m,1} T^*Y \otimes \hat{E}'_k$ for each $k \in \mathbb{N}$. Define $s_k := s'_k - g_k$ for each k, then $\langle s_k, \check{s}_k \rangle$, for $k \in \mathbb{N}$, converges in the sense of distributions to the limit of $\langle s'_k, \check{s}'_k \rangle$ as $k \to \infty$, which is δ_e. Imposing Lemma 4.2, we can achieve that $\sigma = 0$ near D, so that both s_k and $\check{s}_k$ vanish along D for each k. □

References

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