Global Kuranishi charts

Mohan Swaminathan

Symplectothon, November 9–10, 2024

Abstract

These are rough notes for talks at the Southern California Symplectothon (Nov 9–10, 2024) about global Kuranishi charts for Gromov–Witten moduli spaces, following [AMS21] for the genus 0 case and [HS24] for the higher genus case.

Contents

1 Genus 0 case

Our objective is to understand the proof of the following statement.

Theorem 1.1 ([AMS21])**.** Let (X, ω) be a closed symplectic manifold, $A \in H_2(X, \mathbb{Z})$ and $n \geq 0$. Given an ω-tame almost complex structure J, the moduli space $\overline{\mathcal{M}}_{0,n}(X, A; J)$ admits a well-defined equivalence class of oriented global Kuranishi charts having the correct virtual dimension. Moreover, given another ω-tame almost complex structure J', these global Kuranishi charts for $\overline{\mathcal{M}}_{0,n}(X, A; J)$ and $\overline{\mathcal{M}}_{0,n}(X, A; J')$ can be chosen to be oriented cobordant over $\overline{\mathcal{M}}_{0,n} \times X^n$.

We will follow [AMS21, §6]; see also the summary in [HS24, §2.1]. For simplicity, we will ignore marked points and focus on the case $n = 0$ to illustrate the key ideas. The proof will take up a number of steps. Each auxiliary choice we make will be indicated by a bullet point.

1.1 Line bundle on target

Approximate ω in C^∞ to get a symplectic form Ω taming J with $[\Omega] \in H^2(X,\mathbb{Q}) \subset H^2(X,\mathbb{R})$. Multiply Ω by a large integer to assume that $[\Omega]$ lifts to a class in $\gamma \in H^2(X,\mathbb{Z})$. Let $L_\Omega \to X$ be a C^∞ complex line bundle such that $c_1(L_\Omega) = \gamma$.

Lemma 1.2. There is a Hermitian metric $\langle \cdot, \cdot \rangle$ and a compatible Hermitian connection ∇ on L_Ω such that the curvature form of ∇ is given by $-2\pi i \Omega$.

Proof. Choose *any* Hermitian metric $\langle \cdot, \cdot \rangle$ on L_Ω and *any* compatible Hermitian connection ∇' on L_Ω and write the curvature as $-2\pi i \Omega'$. Since γ is a common integral lift of $[\Omega]$ and $[\Omega']$, we can find a smooth (real) 1-form β such that $\Omega' = \Omega + d\beta$. The connection $\nabla = \nabla' + 2\pi i \beta$ now is also Hermitian for $\langle \cdot, \cdot \rangle$ and has curvature given by $-2\pi i \Omega$. □

- Fix the line bundle L_Ω, a metric and a connection on it as in 1.2. Write $d := [\Omega] \cdot A \geq 1$.

1.2 Framed curves

Let $u : \Sigma \to X$ be a J-holomorphic genus 0 stable map with $u_*[\Sigma] = A$.

The line bundle $u^* L_\Omega$, equipped with $(u^*\nabla)^{0,1}$, is a Hermitian holomorphic line bundle whose Chern connection has curvature form $-2\pi i \cdot u^*\Omega$. Since Ω tames J, stability of $u : \Sigma \to X$ implies that $\int u^*\Omega \geq 0$ (resp. > 0) on every irreducible (resp. unstable irreducible) component of Σ. Thus, $u^* L_\Omega$ has degree ≥ 0 on each irreducible component of Σ.

Lemma 1.3. The line bundle $u^* L_\Omega$ is basepoint free, $h^1(\Sigma, u^* L_\Omega) = 0$ and $h^0(\Sigma, u^* L_\Omega) = d + 1$. If $F = (f_0, \ldots, f_d)$ is a $\mathbb{C}$-basis of $H^0(\Sigma, u^* L_\Omega)$, then $\phi_F := [f_0 : \cdots : f_d] : \Sigma \to \mathbb{P}^d$ is a stable map of degree d which is non-degenerate, i.e., not contained in any hyperplane.

Proof. On a nodal curve Σ of genus 0, there is a unique holomorphic line bundle up to isomorphism for each multi-degree. Using this, the result is clear for $\Sigma = \mathbb{P}^1$ since we must have $u^* L_\Omega \simeq \mathcal{O}_{\mathbb{P}^1}(d)$. Use induction on the number of irreducible components to complete the proof. □

Note that there is a $\mathrm{GL}_{d+1}(\mathbb{C})$ worth of choices for F. Later, when we thicken the $\bar\partial$-equation, we will need to break this $\mathrm{GL}_{d+1}(\mathbb{C})$ symmetry and reduce to only a $U(d+1)$ symmetry. For this purpose, it will be useful to distinguish a subclass of 'unitary' F among all possible F.

Lemma 1.4. In 1.3, the $(d+1) \times (d+1)$ Hermitian matrix

$$\mathcal{H}(\Sigma, u, F) := \left(\int_\Sigma \langle f_i, f_j \rangle \, u^*\Omega \right)_{0 \leq i,j \leq d} \tag{1.2.1}$$

is positive definite.

Proof. Indeed, for $0 \neq \mathbf{v} = (v_0, \ldots, v_d)^\top \in \mathbb{C}^{d+1}$, we have

$$\mathbf{v}^\dagger \mathcal{H}(\Sigma, u, F) \mathbf{v} = \int_\Sigma \|v_0 f_0 + \cdots + v_d f_d\|^2 \, u^*\Omega > 0,$$

by unique continuation for $0 \neq v_0 f_0 + \cdots + v_d f_d \in H^0(\Sigma, u^* L_\Omega)$ and the Ω-tameness of J. □

Relaxing J-holomorphicity of u in the above discussion leads to the following notion.

Definition 1.5. A 'framed genus 0 curve' in X of class A is a tuple (Σ, u, F) where

(a) Σ is a nodal genus 0 curve,

(b) $u : \Sigma \to X$ is a smooth map with $u_*[\Sigma] = A$ which is stable, i.e., $\int u^*\Omega \geq 0$ (resp. > 0) on each irreducible (resp. unstable irreducible) component of the curve Σ,

(c) $F = (f_0, \ldots, f_d)$ is a 'framing', i.e., complex basis of the space $H^0(\Sigma, u^*L_\Omega)$ of holomorphic global sections of u^*L_Ω, equipped with the holomorphic structure given by $(u^*\nabla)^{0,1}$, such that the $(d+1)\times(d+1)$ Hermitian matrix $\mathcal{H}(\Sigma, u, F)$ from (1.2.1) is positive definite.

An 'equivalence' of framed genus 0 curves (Σ, u, F) and (Σ', u', F') in X is a biholomorphic map $\varphi : \Sigma \to \Sigma'$ such that $u' \circ \varphi = u$ and $\varphi^*F' = F$. We call a framed genus 0 curve 'unitary' if $\mathcal{H}(\Sigma, u, F)$ is the identity matrix.

To make sense of 1.5(c), note that 1.3 used only the fact that u^*L_Ω has degree ≥ 0 on each irreducible component. Thus, 1.5(b) guarantees that for any framed curve (Σ, u, F), the line bundle u^*L_Ω is automatically basepoint free and we have $h^0(\Sigma, u^*L_\Omega) = d+1$ and $h^1(\Sigma, u^*L_\Omega) = 0$.

Fact. The open sub-orbifold $\overline{\mathcal{M}}^*_{0,0}(\mathbb{P}^d, d) \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d)$, consisting of non-degenerate stable maps, is in fact a smooth variety of the expected dimension. It carries a universal family

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\text{ev}} & \mathbb{P}^d \\ \downarrow{\scriptstyle\pi} & & \\ \overline{\mathcal{M}}^*_{0,0}(\mathbb{P}^d, d) & & \end{array} \tag{1.2.2}$$

whose total space $\mathcal{C}$ is also a smooth variety.

As in 1.3, any framed genus 0 curve (Σ, u, F) gives rise to a non-degenerate degree d map

$$\phi_F = [f_0 : \cdots : f_d] : \Sigma \to \mathbb{P}^d$$

Thus, ϕ_F induces an identification of $\iota_F : \Sigma \hookrightarrow \mathcal{C}$ with a fibre of π in (1.2.2).

1.3 Achieving transversality

- Fix a relatively ample line bundle $\mathcal{L}$ on $\mathcal{C} \to \overline{\mathcal{M}}^*_{0,0}(\mathbb{P}^d, d)$, equipped with a Hermitian metric so that the natural $U(d+1)$ action on $\mathcal{C}$ lifts to a unitary action on the line bundle $\mathcal{L}$.
- Fix a $\mathbb{C}$-linear connection $\nabla^{\mathcal{C}}$ on $T^{*0,1}\mathcal{C}$, which is invariant under the $U(d+1)$ action.
- Fix a $\mathbb{C}$-linear connection ∇^X on TX equipped with J.
- Fix a sufficiently large positive integer $k \gg 1$.

Remark 1.6. We may take $\mathcal{L} = \omega_\pi \otimes \mathrm{ev}^*(\mathcal{O}_{\mathbb{P}^d}(3))$ as in [FP97, §2.3] with a $U(d+1)$-invariant metric obtained by averaging, where ω_π is the relative dualizing line bundle of $\pi : \mathcal{C} \to \overline{\mathcal{M}}^*_{0,0}(\mathbb{P}^d, d)$.

Definition 1.7. We define $\mathcal{K} = (\mathcal{T}, \mathcal{E}, G, \mathfrak{s}, \psi)$ as follows.

(A) The 'thickening' $\mathcal{T}$ is the moduli space of tuples (Σ, u, F, η) where

(i) (Σ, u, F) is a framed genus 0 curve in X of class A and

(ii) $\eta \in E_k(\Sigma, u, F) := H^0(\Sigma, u^*TX \otimes \iota_F^*(T^{*0,1}\mathcal{C} \otimes \mathcal{L}^{\otimes k})) \otimes \overline{H^0(\Sigma, \iota_F^*\mathcal{L}^{\otimes k})}$,

and these satisfy the thickened $\bar{\partial}$-equation

$$\bar{\partial}_J u + \langle \eta \rangle \circ d\tilde{\iota}_F = 0. \tag{1.3.1}$$

Here, the holomorphic structures on $\iota_F^*(T^{*0,1}\mathcal{C})$ and u^*TX come from the connections $\nabla^{\mathcal{C}}$ and ∇^X fixed above, $\tilde{\iota}_F$ is the pullback of $\iota_F : \Sigma \hookrightarrow \mathcal{C}$ to the normalization $\tilde{\Sigma} \to \Sigma$ and the linear map $\langle \cdot \rangle$ is induced by the Hermitian inner product on $\mathcal{L}^{\otimes k}$.

We have a natural projection $\mathcal{T} \to \overline{\mathcal{M}}^*_{0,0}(\mathbb{P}^d, d)$, given by $(\Sigma, u, F, \eta) \mapsto [\phi_F : \Sigma \to \mathbb{P}^d]$.

(B) The 'obstruction bundle' $\mathcal{E} \to \mathcal{T}$ is the vector bundle[1] whose fibre over $(\Sigma, u, F, \eta) \in \mathcal{T}$ is

$$E_k(\Sigma, u, F) \oplus \mathcal{H}_{d+1},$$

with $\mathcal{H}_{d+1}$ being the (real) vector space of $(d+1) \times (d+1)$ Hermitian matrices.

(C) The 'obstruction section' $\mathfrak{s}$ is the section of $\mathcal{E} \to \mathcal{T}$ defined by

$$(\Sigma, u, F, \eta) \mapsto (\eta, \log \mathcal{H}(\Sigma, u, F)),$$

where log denotes the inverse of the exponentiation map from $(d+1) \times (d+1)$ Hermitian matrices to positive definite $(d+1) \times (d+1)$ Hermitian matrices.

(D) The 'symmetry group' $G = U(d+1)$ has a natural action on $\mathcal{T}$, which lifts to $\mathcal{E}$ so that $\mathfrak{s}$ becomes a G-equivariant section.

(E) The 'footprint map'

$$\psi : \mathfrak{s}^{-1}(0)/G \to \overline{\mathcal{M}}_{0,0}(X, A; J)$$

is induced by forgetting the framing.

Claim. In 1.7, the map ψ is a homeomorphism

Proof. Clearly, $\mathfrak{s}^{-1}(0)$ consists of framed curves (Σ, u, F) such that $\bar{\partial}_J u = 0$ and $\mathcal{H}(\Sigma, u, F) = \mathrm{id}$. Also, for $A \in U(d+1)$, we have $\mathcal{H}(\Sigma, u, FA) = A^\dagger \cdot \mathcal{H}(\Sigma, u, F) \cdot A$. From this and 1.4, we conclude that ψ is a bijection. We conclude using the standard fact: a continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism. ∎

[1] We haven't yet explained why this fibrewise description fits into a vector bundle.

Remark 1.8. The compact zero locus $\mathfrak{s}^{-1}(0)$ is completely independent (!) of all the auxiliary choices except for L_Ω with its Hermitian metric and compatible connection.

Fact (Hörmander peak sections trick, [AMS21, 6.24 and 6.26])**.** Let (Σ, u, F) be a framed curve. Define a sequence of finite dimensional vector spaces, indexed by $k \geq 1$, as follows:

$$W_k := \operatorname{image}\left(E_k(\Sigma, u, F) \xrightarrow{\langle\cdot\rangle \circ d\tilde{\iota}_F} \Omega^{0,1}(\tilde{\Sigma}, \tilde{u}^*TX)\right).$$

Then, the subspaces W_k provide an L^2 exhaustion of $\Omega^{0,1}(\tilde{\Sigma}, \tilde{u}^*TX)$ as $k \to \infty$. More precisely, for any $0 \neq \eta \in \Omega^{0,1}(\tilde{\Sigma}, \tilde{u}^*TX)$, we have the following: $\forall\ k \gg 1$, $\exists$ holomorphic sections

$$s_k \in H^0(\Sigma, u^*TX \otimes \iota_F^*(T^{*0,1}\mathcal{C} \otimes \mathcal{L}^{\otimes k})), \quad t_k \in H^0(\Sigma, \iota_F^*\mathcal{L}^{\otimes k})$$

such that the section $\langle s_k, t_k\rangle \circ d\tilde{\iota}_F \in \Omega^{0,1}(\tilde{\Sigma}, \tilde{u}^*TX)$ has a non-trivial L^2 pairing with η.

Corollary 1.9 (Transversality)**.** $\exists\ k \gg 1$ such that for any $(\Sigma, u, F, 0) \in \mathfrak{s}^{-1}(0)$, the linearization

$$\Omega^0(\Sigma, u^*TX) \oplus E_k(\Sigma, u, F) \to \Omega^{0,1}(\tilde{\Sigma}, \tilde{u}^*TX) \tag{1.3.2}$$

of the thickened $\bar{\partial}$-equation (1.3.1) at $(u, 0)$ is surjective and we have

$$H^1(\Sigma, u^*TX \otimes \iota_F^*(T^{*0,1}\mathcal{C} \otimes \mathcal{L}^{\otimes k})) = 0 \quad \text{and} \quad H^1(\Sigma, \iota_F^*\mathcal{L}^{\otimes k}) = 0. \tag{1.3.3}$$

In particular, in a G-invariant neighborhood of $\mathfrak{s}^{-1}(0)$, the projection map $\mathcal{T} \to \overline{\mathcal{M}}^*_{0,0}(\mathbb{P}^d, d)$ is a topological submersion of the expected (relative) dimension and $\mathcal{E} \to \mathcal{T}$ defines a complex vector bundle of the expected rank.

Proof. Given $(\Sigma, u, F, 0) \in \mathfrak{s}^{-1}(0)$, we can find $k \gg 1$ so that (1.3.2) is surjective (by the Hörmander peak sections trick) and (1.3.3) holds (by Serre vanishing [Har77, III.5.2]). Since $\mathfrak{s}^{-1}(0)$ is compact, we can find k which works uniformly; see [HS24, 4.19] for a closely related argument. □

This completes the construction of a global Kuranishi chart for $\overline{\mathcal{M}}_{0,0}(X, A; J)$ associated to the auxiliary data $(L_\Omega, \mathcal{L}, \nabla^{\mathcal{C}}, \nabla^X, k)$.

1.4 Uniqueness

Let $(L_{\Omega_i}, \mathcal{L}_i, \nabla_i^{\mathcal{C}}, \nabla_i^X, k_i)$ be two choices of auxiliary data for $i = 0, 1$.

Going through the above construction, we get integers $d_i := [\Omega_i] \cdot A \geq 1$, finite dimensional vector spaces $E_{k_i}(\Sigma, u, F_i)$ and global Kuranishi charts $\mathcal{K}_i = (\mathcal{T}_i, \mathcal{E}_i, G_i, \mathfrak{s}_i, \psi_i)$ for $i = 0, 1$.

We will show that $\mathcal{K}_0$ and $\mathcal{K}_1$ are equivalent by exhibiting a third global Kuranishi chart $\mathcal{K}_{01}$ which, roughly speaking, 'interpolates' between them. The definition of $\mathcal{K}_{01}$ imitates the usual notion of 'overlap charts' in Kuranishi/implicit atlases.

Definition 1.10. We define $\mathcal{K}_{01} = (\mathcal{T}_{01}, \mathcal{E}_{01}, G_{01}, \mathfrak{s}_{01}, \psi_{01})$ as follows.

(A) The 'double thickening' $\mathcal{T}_{01}$ is the moduli space of tuples $(\Sigma, u, F_0, F_1, \eta_0, \eta_1)$ where

  (i) (Σ, u, F_i) is a framed genus 0 curve in X of class A with respect to $L_{\Omega_i} \to X$,
  (ii) $\eta_i \in E_{k_i}(\Sigma, u, F_i)$

  for $i = 0, 1$ and these satisfy the doubly-thickened $\bar{\partial}$-equation

$$\bar{\partial}_J u + \sum_{i=0,1} \langle \eta_i \rangle \circ d\tilde{\iota}_{F_i} = 0. \tag{1.4.1}$$

(B) $\mathcal{E}_{01} = \mathcal{E}_0 \oplus \mathcal{E}_1 \to \mathcal{T}_{01}$ is the vector bundle with fibre over $(\Sigma, u, F_0, F_1, \eta_0, \eta_1) \in \mathcal{T}_{01}$ given by

$$\bigoplus_{i=0,1} E_{k_i}(\Sigma, u, F_i) \oplus \mathcal{H}_{d_i+1}.$$

(C) The section $\mathfrak{s}_{01} = (\mathfrak{s}_i)_{i=0,1}$ of $\mathcal{E}_{01} \to \mathcal{T}_{01}$ is defined by

$$\mathfrak{s}_i : (\Sigma, u, F_0, F_1, \eta_0, \eta_1) \mapsto (\eta_i, \log \mathcal{H}(\Sigma, u, F_i)).$$

(D) The group $G_{01} = G_0 \times G_1$ is defined by $G_i = U(d_i + 1)$ for $i = 0, 1$.

(E) The map

$$\psi_{01} : \mathfrak{s}_{01}^{-1}(0)/G_{01} \to \overline{\mathcal{M}}_{0,0}(X, A; J)$$

  is induced by forgetting the framings F_i for $i = 0, 1$.

As before, ψ_{01} is a homeomorphism and, in a G_{01}-invariant neighborhood of $\mathfrak{s}_{01}^{-1}(0)$, the space $\mathcal{T}_{01}$ is a manifold of the expected dimension and $\mathcal{E}_{01} \to \mathcal{T}_{01}$ is a vector bundle of the expected rank.

Claim. The global Kuranishi chart $\mathcal{K}_{01}$ is equivalent to $\mathcal{K}_i$ for $i = 0, 1$.

Proof. We suppress the 'germ equivalence' move for global Kuranishi charts from the discussion since we work in a small neighborhood of the zero locus. Observe that $\mathcal{P}_0 = \mathfrak{s}_1^{-1}(0)$ is cut-out transversally near $\mathfrak{s}_{01}^{-1}(0)$. Moreover, the natural projection $\pi_0 : \mathcal{P}_0 \to \mathcal{T}_0$ has the structure of a G_0-equivariant principal G_1-bundle.

Consider the global Kuranishi chart $\mathcal{K}' = (\mathcal{P}_0, \pi_0^*\mathcal{E}_0, \pi_0^*\mathfrak{s}_0, G_{01}, \psi_{01})$.

(I) Applying 'group enlargement' to $\mathcal{K}_0$ (with the group G_1) yields $\mathcal{K}'$.

(II) Applying 'stabilization' to $\mathcal{K}'$ (with the vector bundle $\mathcal{E}_1|_{\mathcal{P}_0}$) yields $\mathcal{K}_{01}$.

This shows that $\mathcal{K}_0$ is equivalent to $\mathcal{K}_{01}$. By symmetry, $\mathcal{K}_1$ is also equivalent to $\mathcal{K}_{01}$. □

Remark 1.11. We omit the very similar proof for the cobordism statement in 1.1.

2 Higher genus case

The following statement is an extension of 1.1 to all genera.

Theorem 2.1 ([HS24], [AMS24])**.** Let (X, ω) be a closed symplectic manifold, $A \in H_2(X, \mathbb{Z})$ and $g, n \geq 0$. Given an ω-tame almost complex structure J, the moduli space $\overline{\mathcal{M}}_{g,n}(X, A; J)$ admits a well-defined equivalence class of oriented global Kuranishi charts of the correct virtual dimension. Moreover, given another ω-tame almost complex structure J', these global Kuranishi charts for $\overline{\mathcal{M}}_{g,n}(X, A; J)$ and $\overline{\mathcal{M}}_{g,n}(X, A; J')$ can be chosen to be oriented cobordant over $\overline{\mathcal{M}}_{g,n} \times X^n$.

We will follow [HS24]; see [AMS24] for an alternate construction. Again, for simplicity, we focus on the case $n = 0$ where we have no marked points. Given that we have already seen the proof of 1.1 in some detail, rather than giving a detailed proof of 2.1, we will only explain the key issues encountered in going to higher genera and how we overcome them.

2.1 Key issues

The following is the main source obstacle to adapting the proof of 1.1 to get 2.1.

Fact. For a nodal curve Σ of genus g, isomorphism classes of topologically trivial holomorphic line bundles on Σ form a complex g-dimensional Lie group $\mathrm{Pic}^0(\Sigma)$, whose tangent space at the identity is $H^1(\Sigma, \mathcal{O}_\Sigma)$. In particular, when $g > 0$, a holomorphic line bundle on Σ is not determined up to isomorphism by its multi-degree.

To elaborate further, consider a J-holomorphic stable map $u : \Sigma \to X$ of genus $g > 0$.

(i) As $u : \Sigma \to X$ varies, the dimension of $H^0(\Sigma, u^* L_\Omega)$ may jump. E.g., this happens when $u : \Sigma \to X$ has a positive genus ghost component.

Remedy. Replace $u^* L_\Omega$ by another natural choice which is (very) ample and has vanishing H^1 on Σ. The standard replacement is $\omega_\Sigma \otimes (u^* L_\Omega)^{\otimes 3}$, or a sufficiently high tensor power of it, where ω_Σ is the dualizing line bundle of Σ; see [FP97, §2.3] or [Sie98]. □

(ii) 'Framing' $u : \Sigma \to X$ using a basis $F = (f_0, \dots, f_N)$ of $H^0(\Sigma, \omega_\Sigma \otimes (u^* L_\Omega)^{\otimes 3})$ produces a non-degenerate stable map $\phi_F = [f_0 : \cdots : f_N] : \Sigma \to \mathbb{P}^N$ such that we have

$$\phi_F^*(\mathcal{O}_{\mathbb{P}^N}(1)) \simeq \omega_\Sigma \otimes (u^* L_\Omega)^{\otimes 3} \tag{2.1.1}$$

as holomorphic line bundles. Deforming ϕ_F among non-degenerate stable maps to $\mathbb{P}^N$ will, in general, disturb (2.1.1). Thus, the space of 'framed genus g curves' in X may not project submersively onto the space of non-degenerate stable maps to $\mathbb{P}^N$.

Remedy. Enlarge the notion of 'framed curve' to allow maps $\phi_F : \Sigma \to \mathbb{P}^N$ for which we only have a topological line bundle isomorphism (2.1.1). This restores submersivity but, to compensate, we must record the difference $[\phi_F^*(\mathcal{O}_{\mathbb{P}^N}(1))] - [\omega_\Sigma \otimes (u^* L_\Omega)^{\otimes 3}] \in \mathrm{Pic}^0(\Sigma)$ as a part of the obstruction section. □

2.2 Picard groups

We follow the exposition in [HS24, Appendix A].

Definition 2.2. We define (relative) Picard groups as follows.

(a) For a nodal curve Σ, define $\mathrm{Pic}(\Sigma)$ to be the group of (isomorphism classes of) holomorphic line bundles on Σ under $\otimes$. Let $\mathrm{Pic}^0(\Sigma)$ be its subgroup of topologically trivial line bundles.

(b) For a family $\pi : C \to S$ of nodal curves, define $\mathrm{Pic}(C/S)$ to be the space of pairs $(s, [\mathcal{L}_s])$, where $s \in S$ and $[\mathcal{L}_s] \in \mathrm{Pic}(C_s)$. Let $\mathrm{Pic}^0(C/S)$ be its subspace where $[\mathcal{L}_s] \in \mathrm{Pic}^0(C_s)$.

2.2.1 Single curve

Fix a nodal curve Σ of genus g. The exponential short exact sequence on Σ is:

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_\Sigma \xrightarrow{\exp} \mathcal{O}_\Sigma^\times \to 0. \tag{2.2.1}$$

Since Σ is connected, applying H^0 to (2.2.1) preserves exactness. Thus, the long exact sequence in cohomology gives the following exact sequence:

$$0 \to H^1(\Sigma, \mathbb{Z}) \xrightarrow{2\pi i} H^1(\Sigma, \mathcal{O}_\Sigma) \xrightarrow{\exp} \mathrm{Pic}(\Sigma) \xrightarrow{c_1} H^2(\Sigma, \mathbb{Z}) \to 0, \tag{2.2.2}$$

using the identification $\mathrm{Pic}(\Sigma) \simeq H^1(\Sigma, \mathcal{O}_\Sigma^\times)$ coming from the Čech description of sheaf cohomology and the vanishing of $H^2(\Sigma, \mathcal{O}_\Sigma)$ coming from Σ being a 1-dimensional scheme. Thus, we have

$$\mathrm{coker}\left(H^1(\Sigma, \mathbb{Z}) \xrightarrow{2\pi i} H^1(\Sigma, \mathcal{O}_\Sigma)\right) \underset{\simeq}{\xrightarrow{\exp}} \mathrm{Pic}^0(\Sigma) := \ker\left(\mathrm{Pic}(\Sigma) \xrightarrow{c_1} H^2(\Sigma, \mathbb{Z})\right). \tag{2.2.3}$$

Fact. The group $\mathrm{Pic}^0(\Sigma)$ is naturally a complex g-dimensional Lie group, with tangent space at the identity being $H^1(\Sigma, \mathcal{O}_\Sigma)$. This follows from the following more precise statement.

(i) If Σ is smooth, then $H^1(\Sigma, \mathbb{Z}) \xrightarrow{2\pi i} H^1(\Sigma, \mathcal{O}_\Sigma)$ is isomorphic to the inclusion of a discrete lattice of rank $2g$ in the vector space $\mathbb{C}^g$. Thus, $\mathrm{Pic}^0(\Sigma)$ is a complex g-dimensional torus.

(ii) In general, let $\Gamma = (V, E)$ be the dual graph of Σ. The vertices $v \in V$ correspond to the connected components $\tilde{\Sigma}_v$ of the normalization $\tilde{\Sigma} \to \Sigma$. The edges $e \in E$ correspond to the unordered pairs of points in $\tilde{\Sigma}$ that get identified to give the nodes $q_e \in \Sigma$.

We then have a natural short exact sequence[2]

$$0 \to H^1(|\Gamma|, \mathbb{C}^\times) \to \mathrm{Pic}^0(\Sigma) \to \prod_{v \in V} \mathrm{Pic}^0(\tilde{\Sigma}_v) \to 0,$$

where $|\Gamma|$ is the geometric realization of Γ viewed as a 1-dimensional simplicial complex.

[2] Its linearization is $0 \to H^1(|\Gamma|, \mathbb{C}) \to H^1(\Sigma, \mathcal{O}_\Sigma) \to \bigoplus_{v \in V} H^1(\tilde{\Sigma}_v, \mathcal{O}_{\tilde{\Sigma}_v}) \to 0$.

2.2.2 Family of curves

Let $\pi : C \to S$ be a holomorphic family of nodal curves of genus g.

Fact (Consequence of [Har77, III.12.11])**.** There is a holomorphic rank g vector bundle

$$\mathbb{H}^*_{C/S} := R^1\pi_*\mathcal{O}_C$$

on S, which is compatible with pulling back families. Its fibre over $s \in S$ is the g-dimensional complex vector space $H^1(C_s, \mathcal{O}_{C_s})$.

Fact (Consequence of [HS24, A.7 and A.9])**.** The family version

$$\exp : \mathbb{H}^*_{C/S} \to \mathrm{Pic}^0(C/S)$$

of the exponential map (2.2.3) is biholomorphic in a neighborhood of the zero section.

Remark 2.3. The full (relative) Picard group $\mathrm{Pic}(C/S) \to S$ is badly behaved: it is non-separated in general. Specifically, the image of the zero section $S \to \mathrm{Pic}(C/S)$, $s \mapsto (s, [\mathcal{O}_{C_s}])$ may not be closed! Luckily, this pathology goes away when we restrict to $\mathrm{Pic}^0(C/S) \to S$.

2.3 Construction

To define a global Kuranishi chart for $\overline{\mathcal{M}}_{g,0}(X, A; J)$, make the following auxiliary choices.

- Choose a line bundle L_Ω with metric and connection as in 1.2.

  Write $d := [\Omega] \cdot A \geq 1$. Write $\mathfrak{L}_u := \omega_\Sigma \otimes (u^*L_\Omega)^{\otimes 3}$ for any stable map $u : \Sigma \to X$ of genus g in class A, with the holomorphic structure induced by the connection pulled back from L_Ω.

- Choose a sufficiently large integer $p \gg 1$ so that for all stable maps $u : \Sigma \to X$ of genus g in class A, the line bundle $\mathfrak{L}_u^{\otimes p}$ is very ample and has vanishing H^1.

  Write $m := p(2g - 2 + 3d)$, $N := m - g$, $\mathcal{G} := \mathrm{PGL}_{N+1}(\mathbb{C})$ and $G := PU(N + 1)$. For a stable map $u : \Sigma \to X$ as above, note that we have $m = \deg(\mathfrak{L}_u^{\otimes p})$ and $N + 1 = h^0(\Sigma, \mathfrak{L}_u^{\otimes p})$.

  Define a 'framed genus g curve' to be a tuple $(\Sigma, u, \mathcal{L}, F)$, where Σ is a nodal genus g curve, $u : \Sigma \to X$ is a stable map as above, $\mathcal{L} \to \Sigma$ is a multi-degree 0 holomorphic line bundle with $H^1(\Sigma, \mathfrak{L}_u^{\otimes p} \otimes \mathcal{L}) = 0$ and $F = (f_0, \ldots, f_N)$ is a complex basis of $H^0(\Sigma, \mathfrak{L}_u^{\otimes p} \otimes \mathcal{L})$.

- Choose a complex linear connection ∇^X on the vector bundle TX equipped with J.

- Choose a $\mathcal{G}$-equivariant map[3] λ from the space of framed genus g curves to $\mathcal{G}/G$.

- Choose a sufficiently large integer $k \gg 1$, which will be used to achieve transversality.

[3] We are suppressing some ugly technicalities here; see [HS24, §3.3] for more details.

Fact. The open sub-stack $\overline{\mathcal{M}}^*_{g,0}(\mathbb{P}^N, m) \subset \overline{\mathcal{M}}_{g,0}(\mathbb{P}^N, m)$, consisting of non-degenerate embeddings $\Sigma \hookrightarrow \mathbb{P}^N$ which also satisfy $H^1(\Sigma, \mathcal{O}_{\mathbb{P}^N}(1)|_\Sigma) = 0$, is in fact a smooth variety of the expected dimension. It carries a universal family

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\text{ev}} & \mathbb{P}^N \\ \downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}}^*_{g,0}(\mathbb{P}^N, m) & & \end{array} \tag{2.3.1}$$

whose total space $\mathcal{C}$ is also a smooth variety.

Any framed genus g curve $(\Sigma, u, \mathcal{L}, F)$ gives a degree m embedding

$$\iota_F = [f_0 : \cdots : f_N] : \Sigma \hookrightarrow \mathbb{P}^N,$$

with $\iota_F^*(\mathcal{O}_{\mathbb{P}^N}(1)) \simeq \mathfrak{L}_u^{\otimes p} \otimes \mathcal{L}$. We have $[\iota_F : \Sigma \hookrightarrow \mathbb{P}^N] \in \overline{\mathcal{M}}^*_{g,0}(\mathbb{P}^N, m)$.

Definition 2.4. We define $(\mathcal{T}, \mathcal{E}, G, \mathfrak{s}, \psi)$ as follows.

(A) The thickening $\mathcal{T}$ is the moduli space of tuples $(\Sigma, u, \mathcal{L}, F, \eta, \alpha)$ where

  (i) $(\Sigma, u, \mathcal{L}, F)$ is a framed genus g curve,
  (ii) $\eta \in E_k(\Sigma, u, \mathcal{L}, F) := H^0(\Sigma, u^*TX \otimes \iota_F^*(T^{*0,1}\mathbb{P}^N \otimes \mathcal{O}_{\mathbb{P}^N}(k))) \otimes \overline{H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k))}$,
  (iii) $\alpha \in H^1(\Sigma, \mathcal{O}_\Sigma)$,

  and these satisfy the thickened $\bar{\partial}$-equation (1.3.1) and the condition $[\mathcal{L}] = \exp(\alpha) \in \mathrm{Pic}^0(\Sigma)$.

  Here, the holomorphic structure on $\iota_F^*(T^{*0,1}\mathbb{P}^N)$ (resp. u^*TX) comes from the isomorphism $T^{*0,1}\mathbb{P}^N \simeq T\mathbb{P}^N$ induced by the Fubini–Study metric (resp. the connection ∇^X).

  The map $\tilde{\iota}_F$ is the pullback of $\iota_F : \Sigma \hookrightarrow \mathbb{P}^N$ to the normalization $\tilde{\Sigma} \to \Sigma$ and the linear map $\langle \cdot \rangle$ is induced by the Hermitian inner product on $\mathcal{O}_{\mathbb{P}^N}(k)$.

  We have a natural projection $\mathcal{T} \to \overline{\mathcal{M}}^*_{g,0}(\mathbb{P}^N, m)$, given by $(\Sigma, u, \mathcal{L}, F, \alpha) \mapsto [\iota_F : \Sigma \hookrightarrow \mathbb{P}^N]$.

(B) The obstruction bundle $\mathcal{E} \to \mathcal{T}$ has fibre over $(\Sigma, u, \mathcal{L}, F, \eta, \alpha)$ given by

$$\mathfrak{su}(N+1) \oplus E_k(\Sigma, u, \mathcal{L}, F) \oplus H^1(C, \mathcal{O}_C).$$

(C) The obstruction section $\mathfrak{s}$ of $\mathcal{E} \to \mathcal{T}$ is given by[4]

$$(\Sigma, u, \mathcal{L}, F, \eta, \alpha) \mapsto (i \log \lambda(\Sigma, u, \mathcal{L}, F), \eta, \alpha).$$

(D) The symmetry group $G = PU(N+1)$.

(E) The footprint map

$$\psi : \mathfrak{s}^{-1}(0)/G \to \overline{\mathcal{M}}_{g,0}(X, A; J)$$

  given by forgetting $\mathcal{L}, F$.

This is the global Kuranishi chart for $\overline{\mathcal{M}}_{g,0}(X, A; J)$ associated to $(L_\Omega, p, \nabla^X, \lambda, k)$. Uniqueness up to equivalence and cobordism are proved as in the genus 0 case.

[4] To define $i \log$, identify $\mathcal{G}/G$ with the space of $(N+1) \times (N+1)$ positive definite Hermitian matrices with $\det = 1$.

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