Kontsevich-Manin axioms

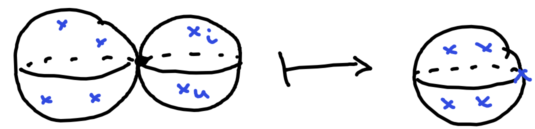
(in genus zero)

I. Introduction

Consider first moduli spaces of stable curves.

There are natural maps

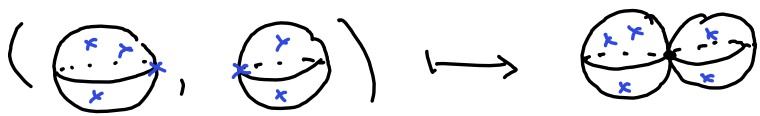
1) $\pi_n : \overline{\mathcal{M}}_{0,n} \longrightarrow \overline{\mathcal{M}}_{0,n-1}$:



forgetful map

2) $S_n \times \overline{\mathcal{M}}_{0,n} \longrightarrow \overline{\mathcal{M}}_{0,n}$ permuting marked points

3) $\varphi : \overline{\mathcal{M}}_{0,n_0+1} \times \overline{\mathcal{M}}_{0,n_1+1} \longrightarrow \overline{\mathcal{M}}_{0,n}$ ← clutching map



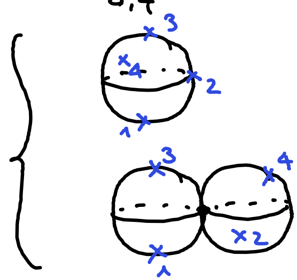
Lem: For each partition $S = S_0 \sqcup S_1$ with $|S_i| \geq 2$ there exists a clutching map φ_S. Its image is a divisor V_S.

Lem: The Poincaré duals $\{\gamma_S := PD(V_S)\}_S$ generate $H^*(\overline{\mathcal{M}}_{0,n};\mathbb{Q})$ as a ring.

Ex: For $n=4$, we have

$$S^2 \xrightarrow{\ \cong\ } \overline{\mathcal{M}}_{0,4}$$

$$z \longmapsto$$



$z \notin \{0, 1, \infty\}$

$z = 1$

Thus, $\gamma_S = \gamma_{S'}$ for any S, S'.

By Talk 7, we can associate to $\overline{\mathcal{M}}^J_{0,0}(X,A)$ a global Kuranishi chart

$$\mathcal{K} = (G, \mathcal{T}, \mathcal{E}, s).$$

Crucially, there exists a submersion

"base space"

$$\mathcal{T} \longrightarrow \underbrace{\mathcal{B}(d)}_{\text{regular, automorphism-free}} \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^d, d) \qquad \text{for some } d \geq 1.$$

Let

$$\pi_X : \overline{\mathcal{M}}_{0,n}(X, A; J) \longrightarrow \overline{\mathcal{M}}_{0,0}(X, A; J)$$

be the forgetful map. Define

$$\mathcal{B}_n(d) := \pi^{-1}(\mathcal{B}(d)) \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^d, d).$$

"nth base space"

Lem: The tuple

$$\mathcal{K}_n := \Big(G, \overbrace{\mathcal{B}_n(d) \times_{\mathcal{B}(d)} \mathcal{T}}^{\mathcal{T}_n}, \overbrace{\mathcal{B}_n(d) \times_{\mathcal{B}(d)} \mathcal{E}}^{\mathcal{E}_n}, \overbrace{\mathrm{id} \times s}^{s_n} \Big)$$

is a global Kuranishi chart for $\overline{\mathcal{M}}_{0,n}(X, A; J)$.

Upshot: We have a virtual fundamental class $[\overline{\mathcal{M}}^J_{0,n}(X,A)]^{vir}$ for any n.

We have an evaluation map

$$ev : \overline{\mathcal{M}}^J_{0,n}(X,A) \longrightarrow X^n$$

and a stabilisation map

$$st : \overline{\mathcal{M}}^J_{0,n}(X,A) \longrightarrow \overline{\mathcal{M}}_{0,n} \qquad (\text{if } n \geq 3)$$

If $n \leq 2$, we take $\overline{\mathcal{M}}_{0,n}$ formally to be a point.

Def: The Gromov-Witten classes of (X,ω) are the homology classes

$$GW^{X,\omega}_{0,n,A} := (ev \times st)_* [\overline{\mathcal{M}}^J_{0,n}(X,A)]^{vir}$$

and its Gromov-Witten invariants are

$$\langle \alpha_1, \dots, \alpha_n ; \sigma \rangle^{X,\omega}_{0,n,A} := \langle \alpha_1 \times \dots \times \alpha_n \times PD(\sigma), GW^{X,\omega}_{0,n,A} \rangle .$$

II. The easy axioms

(Effective) If $\omega(A) < 0$, then $GW^{X,\omega}_{0,n,A} = 0$.

(Grading) $GW^{X,\omega}_{0,n,A}$ has degree $d = \dim_{\mathbb{R}}(X) + 2c_1(A) + 2(n-3)$

(Symmetry) Observe that S_n acts on $\overline{\mathcal{M}}_{0,n}$ by permuting the marked points. Then, we have for any $g \in S_n$ that

$$\langle \alpha_{g(1)}, \dots, \alpha_{g(n)}; g_* \sigma \rangle^{X,\omega}_{0,n,A} = (-1)^{\varepsilon(\alpha, g)} \langle \alpha_1, \dots, \alpha_n; \sigma \rangle^{X,\omega}_{0,n,A}$$

$\rightsquigarrow$ VFC is S_n-invariant

(Zero) If $A = 0$, then

$$\langle \alpha_1, \dots, \alpha_n; \sigma \rangle^{X,\omega}_{0,n,A} = \begin{cases} b \cdot \langle \alpha_1 \cup \dots \cup \alpha_n, [X] \rangle & \sigma = b \cdot [pt] \\ 0 & \text{otherwise} \end{cases}$$

$\rightsquigarrow$ $\overline{\mathcal{M}}_{0,n}(X, 0; J) = X \times \overline{\mathcal{M}}_{0,n}$ is regular $\Longrightarrow$ VFC = FC

III. Fundamental class axiom

Let $\pi_n : \overline{\mathcal{M}}_{0,n}(X,A;J) \to \overline{\mathcal{M}}_{0,n-1}(X,A;J)$ just forget the nth marked point.

(Fundamental class) If $n \geq 1$, then

$$\langle \alpha_1, \dots, \alpha_{n-1}, 1; \sigma \rangle^{X,\omega}_{0,n,A} = \langle \alpha_1, \dots, \alpha_{n-1}; (\pi_n)_* \sigma \rangle^{X,\omega}_{0,n-1,A}$$

Intuition: There is no constraint on the nth marked point.

In terms of VFC's, this is equivalent to

$$(\pi_n)_* \left(st^* PD(\sigma) \cap [\overline{\mathcal{M}}_{0,n}(X,A;J)]^{vir} \right) = st^* PD(\pi_{n*}\sigma) \cap [\overline{\mathcal{M}}_{0,n-1}(X,A;J)]^{vir}$$

If π_n and st were submersions, this would follow from general algebraic topology. Given a GKC, the proof is essentially the same.

IV. Splitting axiom

(Splitting) Write $PD(\Delta_X) = \sum_i \beta_i \times \gamma_i$ and let

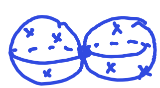
$$\varphi : \overline{\mathcal{M}}_{0,n_0+1} \times \overline{\mathcal{M}}_{0,n_1+1} \longrightarrow \overline{\mathcal{M}}_{0,n}$$

be a clutching map. Then

$$\langle \alpha_1, \dots, \alpha_n ; \varphi_*(\sigma_0 \times \sigma_1) \rangle^{X,\omega}_{0,n,A}$$

$$= \sum_{A_0 + A_1 = A} \sum_i \langle \alpha_1, \dots, \alpha_{n_0}, \beta_i ; \sigma_0 \rangle^{X,\omega}_{0,n_0+1,A_0} \cdot \langle \gamma_i, \alpha_{n_0+1}, \dots, \alpha_n ; \sigma_1 \rangle^{X,\omega}_{0,n_1+1,A_1}$$

Intuition: Suppose $\sigma_i = [\overline{\mathcal{M}}_{0,n_i+1}]$. Then

LHS $= \# \{$  $\to X \} = \# \{$  $\to X \} \times \{$  $\to X \}$

$= $ RHS

Proof sketch: The map φ usually does not lift to a map

$$\underbrace{\mathcal{K}_{A_0, n_0+1} \times_X \mathcal{K}_{A_1, n_1+1}}_{(*)} \longrightarrow \mathcal{K}_{A, n}$$

Instead: new GKC for $\overline{\mathcal{M}}(A_0, A_1) := \overline{\mathcal{M}}_{0, n_0+1}(X, A_0; J) \times_X \overline{\mathcal{M}}_{0, n_1+1}(X, A_1; J)$.

Set

$$\underbrace{B_{n_0, n_1}(d_0, d_1)}_{\text{regular!}} := \overline{\mathcal{M}}_{0, n_0+1}(\mathbb{P}^d, d_0) \times_{\mathbb{P}^d} \overline{\mathcal{M}}_{0, n_1+1}(\mathbb{P}^d, d_1) \cap \varphi^{-1}(B_n(d))$$

Then

$$\mathcal{T}_{A_0, A_1} := B_{n_0, n_1}(d_0, d_1) \times_{B_n(d)} \mathcal{T}$$

is a manifold with $\tilde{\varphi} : \mathcal{T}_{A_0, A_1} \longrightarrow \mathcal{T}_n$.

Lem: $\mathcal{K}_{A_0, A_1} := (G, \mathcal{T}_{A_0, A_1}, \tilde{\varphi}^* \mathcal{E}_n, \tilde{\varphi}^* s_n)$ is a GKC for $\overline{\mathcal{M}}(A_0, A_1)$, equivalent to $(*)$.

V Divisor axiom

(Divisor) Suppose $n \geq 1$ and $|\alpha_n| = 2$. Then

$$\langle \alpha_1, \dots, \alpha_n; [\overline{\mathcal{M}}_{0,n}] \rangle^{X,\omega}_{0,n,A} = \langle \alpha_n, A \rangle \cdot \langle \alpha_1, \dots, \alpha_{n-1}; [\overline{\mathcal{M}}_{0,n-1}] \rangle^{X,\omega}_{0,n-1,A}$$

Intuition: If $\alpha_n = PD(Y)$, $Y \subset X$ divisor, a generic curve $u: S^2 \to X$ has

$$Y \cdot u = \langle \alpha_n, A \rangle$$

$\Longrightarrow \exists$ "$\langle \alpha_n, A \rangle$ many places" for the nth marked point.

Proof sketch: 1) We can arrange for $ev_n: \mathcal{T}_n \to X$ to be a submersion.

2) $\mathcal{T}_Y := ev_n^{-1}(Y)$ is a manifold with

$$\dim(\mathcal{T}_Y) = \dim(\mathcal{T}_{n-1}).$$

3) The vfc of $(G, \mathcal{T}_Y, \mathcal{E}_n|_{\mathcal{T}_Y}, s_n|_{\mathcal{T}_Y})$ is

$$ev_n^* \alpha_n \cap [\overline{\mathcal{M}}_{0,n}(X, A; J)]^{vir}$$

4) $\deg(\pi_n|_{\mathcal{T}_Y}) = \langle \alpha_n, A \rangle$.

Thank you

for

your attention !