Syllabus for Fall 2022 Southern California Symplectothon

Contents

1. Overview

Our goal is to learn some of techniques surrounding Lagrangian Floer theory in symmetric products, which features prominently in the recent papers [MS, PS, CGHM$^+$] among others. In particular, we aim to build up Lagrangian Floer theory in a very general context involving symplectic orbifolds and bulk deformed superpotentials. We will briefly discuss previous nondisplaceability results for Lagrangians in toric symplectic manifolds and orbifolds by Fukaya–Oh–Ohta–Ono, Cho–Poddar, and others. We will then discuss recent computational advances for Lagrangian Floer homology in symmetric products (with natural comparisons to Heegaard Floer theory) and some of their applications. Lastly, we discuss Lagrangian spectral invariants in the context of symmetric products and implications for quantitative symplectic geometry.

2. List of talks

2.1. Talk 1: Overview (Kyler Siegel).

Description. Give a high level overview of the main results we will be discussing. In particular, introduce some of the results in the primary three references: [MS, PS, CGHM$^+$], along with brief sketches of the techniques involved.

Some topics to cover.

- nondisplaceable links in $S^2 \times S^2$ for which each component is individually displaceable
- Polterovich's lemma on stable displacements via suspension (see [MS, Lem. 1.11])
- brief comparison to other nondisplaceability results due to Fukaya–Oh–Ohta–Ono and others
- brief proof outline of main result in [MS] and key elements involved (symmetric products, orbifold Floer homology, bulk deformed superpotentials, etc)
- link spectral invariants and some of their main properties
- discuss some of the applications from [PS, CGHM$^+$] and comparison of techniques involved.

Possible reference(s).

- Primary: [MS, PS, CGHM$^+$]
- Secondary:

2.2. Talk 2: Lagrangian Floer theory and spectral invariants (Siyang Liu).

Description. Give a crash course in Lagrangian Floer theory, illustrated with some simple examples, and discuss the definition of Lagrangian spectral invariants.

Some topics to cover.

- recall the basics of Lagrangian Floer homology for a monotone Lagrangian
- the example of an equator in S^2, and possibly a product of two equators in $S^2 \times S^2$
- define spectral invariants via Lagrangian Floer homology.

Possible reference(s).

- Primary: [LZ]
- Secondary: [Oh, §16]

2.3. Talk 3: Symmetric products, the tautological correspondence, and Heegaard Floer theory (Roman Krutowski).

Description.

Some topics to cover.

- define symmetric products of symplectic manifolds
- describe the tautological correspondence for curves in symmetric products
- briefly give the definition of Heegaard Floer theory for links in three-manifolds (c.f. [CGHM$^+$, Rmk. 4.1, Rmk. 4.2]).

Possible reference(s).

- Primary: [MS, §3.1]
- Secondary: [Lip], [OS]

2.4. Talk 4: Advanced Lagrangian Floer theory (Sheel Ganatra).

Description. Describe the curved $\mathcal{A}_\infty$ algebra associated to a Lagrangian, and relevant notions such as bounding cochains and obstructedness. Discuss the superpotential and its critical points, possibly with bulk deformations, and implications for nondisplaceability.

Some topics to cover.

- the (curved) Fukaya $\mathcal{A}_\infty$ algebra and bounding cochains
- critical points of the critical points
- bulk deformations
- applications to nondisplaceability
- possibly a few words on the case of orbifold singularities.

Possible reference(s).

- Primary: [MS, §2]
- Secondary: [FOOO1]

2.5. Talk 5: Nondisplaceability results for Lagrangians in toric symplectic manifolds (Chris Kuo).

Description. Discuss what is known about nondisplaceable Lagrangians in compact toric symplectic manifolds.

Some topics to cover.

- possibly some simple examples such as S^2 or $\mathbb{CP}^2$
- every compact toric manifold has a nondisplaceable fiber (see [FOOO2, Thm 1.5]
- nondisplaceable fibers via bulk deformations (see [FOOO4])
- continuum of nondisplaceable tori in $S^2 \times S^2$ (see [FOOO5]).

Possible reference(s).

- Primary: [FOOO2, FOOO4]
- Secondary: [FOOO3]

2.6. Talk 6: Symplectic orbifolds (Joj Helfer).

Description. Introduce symplectic orbifolds and their pseudoholomorphic curve theory.

Some topics to cover.

- basics of orbifolds
- symplectic structures, compactible almost complex structures
- inertia orbifolds
- basics of orbifold Gromov–Witten theory
- holomorphic disk counting in symplectic orbifolds.

Possible reference(s).

- Primary: [CR]
- Secondary: [ALR, CP]

2.7. Talk 7: Superpotential calculations in symmetric products (Cheuk Yu Mak).

Description. Discuss Mak–Smith's superpotential computation in symmetric products, and use it to conclude their proof of nondisplaceability for links in $S^2 \times S^2$.

Some topics to cover.

- recall the desired formula for the superpotential, and explain how this can be used to establish nondisplaceability
- explain the heuristic computation of in terms tropical pictures, and also explicit curve constructions.

Possible reference(s).

- Primary: [MS]
- Secondary:

2.8. Talk 8: Lagrangian spectral invariants in symmetric products and applications (Egor Shelukhin).

Description. Discuss spectral invariants in symmetric product orbifolds and some of their applications, with particular focus on $S^2 \times S^2$.

Some topics to cover.

- briefly recall the construction of Lagrangian spectral invariants in a general setting
- Lagrangian estimators and their main properties
- applications to Hofer's geometry and beyond.

Possible reference(s).

- Primary: [PS]
- Secondary:

2.9. Talk 9: Quantitative Heegaard Floer theory (Ivan Smith).

Description. Discuss quantitive Heegaard Floer theory and link spectral invariants.

Some topics to cover.

- the technical setup in [CGHM$^+$] and why orbifolds can be avoided
- the relationship between Heegaard Floer homology for links in three-manifolds and link spectral invariants
- properties of link spectral invariants and sample applications.

Possible reference(s).

- Primary: [CGHM$^+$]
- Secondary:

References

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[OS] Peter Ozsváth and Zoltán Szabó. Holomorphic disks, link invariants and the multi-variable Alexander polynomial. *Algebraic & Geometric Topology* **8**(2008), 615–692.

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