A Symplectic Conspectus

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This talk is intended to be an introduction to Fukaya categories and their applications for graduate students with varying amounts of familiarity with symplectic topology. We will present some of the key concepts in this rapidly developing field while ignoring most of the grief-inducing technical details. Our main references are the excellent survey articles [Aur] and [Smi], whose bibliographies cover all the results in this talk.

1 Some questions

Question: What are the Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega_{\text{std}})$?

- *n=1:* Any embedded $S^1 \subset \mathbb{R}^2$ is Lagrangian.
- *n=2:* An orientable Lagrangian in $\mathbb{R}^4$ must be a torus. The non-orientable ones are precisely those with $\chi \equiv 0 \bmod 4$ and $\chi \neq 0$.
- *n=3:* A prime Lagrangian three-manifold in $\mathbb{R}^6$ must be $S^1 \times \Sigma_g$. For any orientable Y^3, $Y^3 \# S^1 \times S^2$ embeds as a Lagrangian in $\mathbb{R}^6$. It is unknown which connected sums of three manifolds embed as Lagrangians (for example connected sums of hyperbolic three-manifolds).

Gromov proved that there are no *exact* Lagrangians in $\mathbb{R}^{2n}$ (i.e. $\lambda_{\text{std}} = -\sum y_i dx_i$ restricts to an exact one-form). This result has subsequently been generalized to a large class of ambient symplectic manifolds.

After exact Lagrangians, the next best class of Lagrangians are the *monotone* Lagrangians, meaning that the symplectic area and Maslov homomorphisms $\pi_2(\mathbb{R}^{2n}, L) \to \mathbb{R}$ are positively proportional. Up to *Hamiltonian isotopy*, we have:

- *n=1:* Two Lagrangians $S^1 \subset \mathbb{R}^2$ are Hamiltonian isotopic if and only if they bound the area same.
- *n=2:* There are at least two distinct monotone Lagrangian tori in $\mathbb{R}^4$, the so called Clifford and Chekanov tori.
- *n=3: Auroux '14:* there are infinitely many distinct monotone Lagrangian tori in $\mathbb{R}^{2n \geq 6}$.

Remark 1.1 *Every symplectic manifold is locally symplectomorphic to* $\mathbb{R}^{2n}$*, hence the Lagrangian classification problem is at least as hard as this one!*

Another very important class of symplectic manifolds are cotangent bundles, which inherit canonical symplectic structures. Arnold's famous conjecture is still open:

Conjecture 1.2 *(Arnold) Every closed* exact *Lagrangian in* T^*M *(*M *any smooth manifold) is Hamiltonian isotopic to the* 0*-section.*

Corollary 1.3 *Two smooth manifolds are diffeomorphic if and only if their cotangent bundles are symplectomorphic.*

The current state of the art is that for any exact Lagrangian $L \subset T^*M$, the restriction $\pi|_L \to M$ is a homotopy equivalence.

2 Algebraic formalism

- For Lagrangians $L, L' \subset (M, \omega)$, we associate to them $HF(L, L')$, a $\mathbb{Z}$-graded $\mathbb{K}$-vector space in the best cases (here $\mathbb{K}$ is our chosen ground field).
- HF categorifies the homological intersection number:

$$\sum_i (-1)^i \dim HF^i(L, L') = \pm[L] \cdot [L'].$$

- $HF(L, L')$ is invariant under Hamiltonian isotopies of L and L'
- *Floer:* If $\langle \omega, \pi_2(M, L) \rangle = 0$, $HF^*(L, L) \cong H^*(L)$.

Definition 2.1 *The Donaldson category* $Don(M, \omega)$ *has objects Lagrangians in* (M, ω) *and* $Mor(L, L') := HF(L, L')$.

Remark 2.2 *This category is useful, but lots of subtle chain level information has been thrown away by passing to (co)homology.*

There are also higher operations:

$$\mu^k : CF(L_{k-1}, L_k) \otimes ... \otimes CF(L_0, L_1) \to CF(L_0, L_k).$$

For intersection points $p_1 \in CF(L_0, L_1), ..., p_k \in CF(L_{k-1}, L_k)$, by definition $\mu^k(p_k, ..., p_1) = \sum_{q \in L_0 \cap L_k} nq$, where n is the count of pseudoholomorphic $(k+1)$-gons with sides mapping to $L_0, L_1, ..., L_k$ respectively and vertices mapping to $q, p_1, p_2, ..., p_k$ respectively. Together these higher operations satisfy the $\mathcal{A}_\infty$ relations. This follows by Gromov's compactness theorem and studying various ones in which a pseudoholomorphic polygon can degenerate. The first relation amounts to the fact that μ^1 squares to zero, the second says that μ^2 satisfies a Leibnitz rule with respect to μ^1, the third gives a precise sense in which μ^2 fails to be an associative product, and the higher relations give higher order analogs of non-associativity.

Remark 2.3

- μ^2 *descends to a product*

$$HF(L_1, L_2) \otimes HF(L_0, L_1) \to HF(L_0, L_2).$$

  In particular, $HF(L, L)$ *is a unital ring.*

- $\mu^{k \geq 3}$ *does* not *descend to cohomology.*

Since the μ^k satisfy the $\mathcal{A}_\infty$ relations, we can make the following definition.

Definition 2.4 *Let the Fukaya category Fuk*(M, ω) *be the* $\mathcal{A}_\infty$ *category with objects Lagrangians in* M *and Mor*$(L, L') := CF(L, L')$*, along with the higher operations* $\mu^k : Mor(L_{k-1}, L_k) \otimes \ldots \otimes Mor(L_0, L_1) \to Mor(L_0, L_k)$.

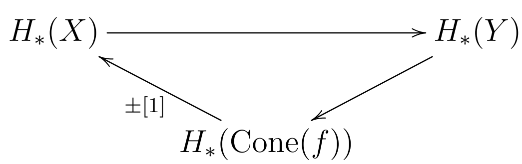
3 Exact sequences

3.1 Triangulated $\mathcal{A}_\infty$ categories

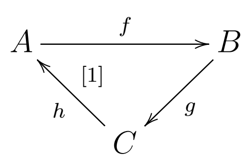
Recall that for $f : X \to Y$ a map of topological spaces, we can form

$$\text{Cone}(f) = X \times I \cup Y / ((p, 1) \sim f(p)),$$

and there is an associated exact triangle



For X, Y objects in an $\mathcal{A}_\infty$ category $\mathcal{A}$ and $f : X \to Y$ a *closed* morphism, we can sometimes complete f to a triangle



such that we get natural long exact sequences with respect to any test object T:

$$\ldots \longrightarrow H^i \hom(T, A) \xrightarrow{f} H^i \hom(T, B) \longrightarrow H^i \hom(T, C) \longrightarrow H^{i+1} \hom(T, A) \longrightarrow \ldots$$

In fact, there is a notion of C being a *mapping cone* for $f : A \to B$ and in this case the triangle of morphisms f, g, h is called an *exact triangle.* We say that $\mathcal{A}$ is *triangulated* if mapping cones always exist.

Unfortunately, Fuk(M,ω) is typically *not* triangulated. Luckily, there's a nice fix:

$$Fuk(M,\omega) \rightsquigarrow TwFuk(M,\omega) \rightsquigarrow DFuk(M,\omega)$$

We first form the category $TwFuk(M,\omega)$, whose objects are twisted complexes. One can think of a twisted complex as the $\mathcal{A}_\infty$ version of a chain complex. We then pass to the cohomology category (i.e. keep the same objects but declare the morphism space between any two objects to be the cohomology of the corresponding chain complex) to form the *derived Fukaya category* $DFuk(M,\omega)$. The derived Fukaya category is an honest category (as opposed to $\mathcal{A}_\infty$). Note: there is sometimes one additional step involving splitting idempotents which we are ignoring here for brevity.

Remark 3.1 *One huge motivation to study $DFuk$ (outside the scope of this talk) is that it is half of homological mirror symmetry. Namely, HMS claims it should be equivalent to a derived category of coherent sheaves on the mirror manifold of M.*

3.2 Symplectic mapping class groups and Dehn twists

Consider the fibration $\pi : \mathbb{C}^{n+1} \to \mathbb{C}$ which sends $(z_1, ..., z_{n+1})$ to $\sum z_j^2$.

- A generic fiber (i.e. $\pi^{-1}(x), x \neq 0$) is symplectomorphic to $(T^*S^n, \omega_{\text{can}})$.
- The monodromy around $0 \in \mathbb{C}$ is a compactly supported symplectomorphism τ of T^*S^n, usually called a "Dehn twist" (well-defined up to symplectic isotopic), which restricts to the antipodal map on S^n.

Fact: For n *even*, τ has order two as a smooth diffeomorphism (at least for $n = 2, 6$) but infinite order as a symplectomorphism! Indeed, one can show that $HF^*(\tau^m(T_p^*), T_{p'}^*)$ grows arbitarily with m (here T_p^* and $T_{p'}^*$ are two random cotangent fibers).

Remark 3.2 *Given any Lagrangian sphere $S^n \subset (M^{2n}, \omega)$, S^n has a "Weinstein neighborhood" $U \supset S$ which is symplectomorphic to a neighborhood of S^n in T^*S^n. Hence one can make sense of the "Dehn twist around S^n in M^{2n}".*

Theorem 3.3 *(Seidel) Given a Lagrangian sphere $S^n \subset (M^{2n}, \omega)$ and $L \in Fuk(M,\omega)$ any object, there is an exact triangle in $TwFuk(M,\omega)$ of the form*

$$\begin{array}{ccc} HF^*(S,L) \otimes S & \xrightarrow{ev} & L \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & \tau_S(L) & \end{array}$$

Note: the first term doesn't make sense as a "geometric" Lagrangian, but such a formal sum of shifted copies of S is perfectly allowed in $TwFuk(M,\omega)$!

Corollary 3.4 *For any test object T we get a long exact sequence of Floer homology groups:*

$$HF^*(S,L) \otimes HF^*(T,S) \xrightarrow{\mu^2} HF^*(T,L) \longrightarrow HF^*(T,\tau_S(L)) \xrightarrow{[1]} ...$$

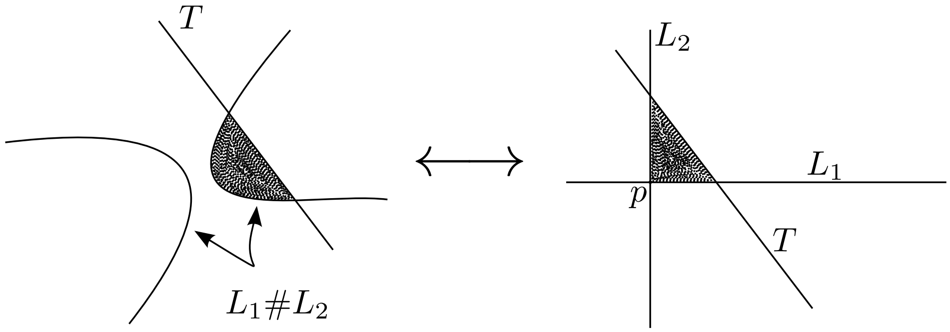
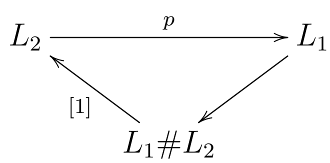


Figure 1: Relating holomorphic strips after surgery to holomorphic triangles before surgery.

3.3 Lagrange surgery

Another important source of exact triangles comes from Lagrange surgery. Namely, suppose L_1, L_2 are Lagrangians (not necessarily spheres!) intersecting transversely at exactly one point. There is a well-defined way to resolve the interestion point to form a smooth Lagrangian $L_1 \# L_2$, called the *Lagrange surgery* of L_1 and L_2. This fits into an exact triangle with L_1 and L_2 of the form



The content of this is roughly that holomorphic strips between $L_1 \# L_2$ and some test Lagrangian T correspond to holomorphic triangles between L_1, L_2, and T, as in Figure 1.

Remark 3.5 *If L_2 is a sphere, $L_1 \# L_2$ is Hamiltonian isotopic to $\tau_{L_2}(L_1)$ and we recover Seidel's exact triangle.*

4 Generation and computations

In order to actually "compute" a Fukaya category, it's useful to have a precise sense in which a small collection of objects determine the entire category. Let $\mathcal{A}$ be an $\mathcal{A}_\infty$ category.

Definition 4.1

- $G_1, ..., G_r \in Obj(\mathcal{A})$ generate $\mathcal{A}$ *if (in $Tw\mathcal{A}$) every object of $\mathcal{A}$ is built from copies of $G_1, ..., G_r$ (equivalently, every object in $\mathcal{A}$ is an iterated mapping cone of $G_1, ..., G_r$).*

- $G_1, ..., G_r$ split-generate $\mathcal{A}$ *if every object of $\mathcal{A}$ is quasi-isomorphic to a direct summand of a twisted complex built from $G_1, ..., G_r$.*

These notations are perhaps best illustrated with an example.

Example 4.2 *Consider the standard torus $\mathbb{T}^2$ with the standard Lagrangian circles α, β which generate $H_1(\mathbb{T}^2)$. Taking iterating mapping cones of α and β, we can generate a curve in possible isotopy class of simple closed curves in $\mathbb{T}^2$. For example, the resolution $\beta\#\alpha \simeq \tau_\alpha\beta$ lies in the homology class $[\beta] \pm [\alpha]$. One might therefore guess that α and β generate $Fuk(\mathbb{T}^2)$.*

However, let $\theta \in \Omega^1(\mathbb{T}^2 \setminus \{pt\})$ satisfy $d\theta = \omega$ and $\int_\alpha \theta = \int_\beta \theta = 0$. Then one can check that the integral of θ over any iterated mapping cone of α and β also vanishes. This means that α and β only generate the subcategory of $Fuk(\mathbb{T}^2)$ consisting of those Lagrangians which are "balanced" with respect to θ.

Now let $\gamma \subset \mathbb{T}^2$ be the circle with slope $-1/2$. Note that γ intersects β in two points, say q_1 and q_2. We can therefore consider the morphism $T^{a_1}q_1 + T^{a_2}q_2 \in Hom(\gamma, \beta)$. Here T is the formal Novikov variable usually required to make Floer homology well-defined. The mapping cone of $T^{a_1}q_1 + T^{a_2}q_2$ is the resolution of $\beta \cap \gamma$ and consists of two components, whose Hamiltonian isotopy classes depend on which numbers a_1 and a_2 we pick. These components appear by definition in the category split-generated by β and γ. Actually γ is in the category generated by α and β, and one can check that in fact α and β split-generate *$Fuk(\mathbb{T}^2)$, i.e. all Hamiltonian isotopy classes of simple closed curves can be obtained via this procedure.*

We conlude the talk by stating a few big results.

Theorem 4.3 *Every compact exact Lagrangians in T^*X is isomorphic in $Fuk(T^*X)$ to the* 0*-section.*

Consider the n-dimensional Milnor fiber A_2^n, i.e. the Stein manifold obtained by plumbing together two copies of T^*S^n. Let L_1 and L_2 be the two corresponding 0-sections.

Theorem 4.4 *Every exact Lagrangian with zero Maslov class in $A_2^{n\geq 3}$ is in the subcategory generated by L_1 and L_2.*

Corollary 4.5 *Every exact Lagrangian in A_2^n with zero Maslov class is a homology sphere.*

References

[Aur] Denis Auroux. A Beginner's introduction to Fukaya categories. *arXiv:1301.7056* (2013).

[Smi] Ivan Smith. A symplectic prolegomenon. *arXiv:1401.0269* (2014).