The Space of Negative Scalar Curvature Metrics

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November 12, 2012

1 Introduction

Review: So far we've been considering the question of which manifolds admit positive scalar curvature (p.s.c.) metrics. On the one hand, we've mentioned that there are obstructions to admitting p.s.c. metrics, for example coming from the index theory of dirac operators for spin manifolds. On the other hand, we saw last time that the condition of admitting a p.s.c. metric is preserved under surgeries of codimension ≥ 3, and this allowed us to conclude that lots of manifolds admit p.s.c. metrics. We now want to consider the question at one meta level higher, namely "what is the topology of the space of p.s.c. metrics?".

Actually, this topology can be quite complicated in general, as the following theorem illustrates. In what follows, let $\mathcal{R}, \mathcal{R}^+, \mathcal{R}^-$ denote the space of metrics with any scalar curvature, (strictly) positive scalar curvature, and (strictly) negative scalar curvature respectively.

Theorem 1.1 *[LM] Let X be a closed spin manifold of dimension n with $\mathcal{R}^+(X) \neq \varnothing$. Then*

- $\pi_0(\mathcal{R}^+(X)) \neq \varnothing$ *if* $n \equiv 0, 1$ *(mod 8)*
- $\pi_1(\mathcal{R}^+(X)) \neq \varnothing$ *if* $n \equiv 0, -1$ *(mod 8)*
- $\pi_0(\mathcal{R}^+(X))$ *is infinite if* $n = 4k - 1 \geq 7$ *for some* $k \in \mathbb{N}$
- $\pi_0(\mathcal{R}^+(S^7)/\mathit{Diff}(S^7))$ *is infinite*

In this talk we'll consider the related but much easier question: "what is the topology of $\mathcal{R}^-(M)$ for a closed manifold M?"

Remark 1.2 *By the Kazdan-Warner trichotomy, we know that $\mathcal{R}^-(M)$ is always non-empty.*

Here is our main result [Loh]:

Theorem 1.3 *(Lohkamp) For M a closed manifold of dimension $n \geq 3$, $\pi_i(\mathcal{R}^-(M)) = 0$ for $i \geq 0$.*

Using general results on infinite dimensional manifolds, we get

Corollary 1.4 *(Palais-Whitehead) [Pal]* $\mathcal{R}^-(M)$ *is contractible.*

Remark 1.5 *Here we are endowing* $\mathcal{R}^-(M)$ *with the* $\mathcal{C}^\infty$ *topology. Loosely speaking, two metrics are close if, which we write down their expressions in local coordinates, their derivatives of all orders are close. There is more or less a unique reasonable way to formalize this when* M *is compact. However, we won't be using any subtle properties of the* $\mathcal{C}^\infty$ *topology here.*

2 The Yamabe Problem

We now embark on a brief tangent to discus the Yamabe problem.

Remark 2.1 *The Yamabe problem has a long and interesting history, which we won't discuss here.*

Let $\mathrm{Diff}^+(M)$ act on $\mathcal{C}^\infty(M)$ by precomposition, giving rise to the semi-direct product $\mathcal{C}^\infty(M) \rtimes \mathrm{Diff}^+(M)$. Recall the uniformization theorem (see [RS] for a brief summary)

Theorem 2.2 *(Uniformization) For* M *an oriented connected closed* 2*-manifold,* $\mathcal{C}^\infty(M) \rtimes$ *Diff*$^+(M)$ *acts transitively on* $\mathcal{R}(M)$*.*

Corollary 2.3 *Every metric on* M *is conformally equivalent to a metric of constant (sectional) curvature.*

Remark 2.4 *The obvious generalization to higher dimensions is too much to ask for, since the full curvature tensor has too many degrees of freedom. However, we do have:*

Theorem 2.5 *(Yamabe Problem)(Aubin,Schoen,Trudinger,Yamabe) [LP] For* (M, g) *a closed Riemannian manifold of dimension* $n \geq 3$*, there exists a conformally equivalent metric of constant scalar curvature.*

Why is this relevant to our problem? First observe that the condition $\pi_i(\mathcal{R}^-(M)) = 0$ means that any map $f : S^i \to \mathcal{R}^-(M)$ has an extension to the ball $F : B^{i+1} \to \mathcal{R}^-(M)$.

$$\begin{array}{ccc} S^i & \xrightarrow{\ f\ } & \mathcal{R}^-(M) \\ \big\downarrow & \overset{F}{\nearrow} & \\ B^{i+1} & & \end{array}$$

Our strategy will be as follows:

- **Step 1:** Extend f to a map $F_1 : B^{i+1} \to \mathcal{R}(M)$ using convexity of $\mathcal{R}(M)$.

$$\begin{array}{ccc} S^i & \xrightarrow{\quad f \quad} & \mathcal{R}(M) \\ \big\downarrow & \overset{F_1}{\nearrow} & \\ B^{i+1} & & \end{array}$$

- **Step 2:** By connect-summing (in a continuous way) with a copy of S^n with highly negative scalar curvature, alter F_1 to an extension of f of the form $F_2 : B^{i+1} \to \mathcal{R}^-_{av}(M)$. Here $\mathcal{R}^-_{av}(M)$ denotes the space of metrics g on M such that the average scalar curvature

$$\int_M R_g dV_g < 0,$$

where R_g denotes the scalar curvature associated to g.

$$\begin{array}{ccc} S^i & \xrightarrow{\quad f \quad} & \mathcal{R}^-_{av}(M) \\ \big\downarrow & \overset{F_2}{\nearrow} & \\ B^{i+1} & & \end{array}$$

- **Step 3:** Conformally rescale the metrics corresponding to F_2 with appropriate functions on M, to obtain a new map F such that the resulting metrics have honest negative scalar curvature.

$$\begin{array}{ccc} S^i & \xrightarrow{\quad f \quad} & \mathcal{R}^-(M) \\ \big\downarrow & \overset{F}{\nearrow} & \\ B^{i+1} & & \end{array}$$

Remark 2.6 *It is this last step that shares close connections with the Yamabe problem. In order to find an appropriate scaling function to take us from average negative scalar curvature to actual negative scalar curvature, we'll need to analyze a similiar PDE to that of the Yamabe problem.*

3 Warm Up

As a warm up, we prove the following:

Proposition 3.1 *[RS] $\mathcal{R}^+(S^2)$ is contractible.*

Proof By uniformization, $\mathcal{C}^\infty(S^2) \rtimes \mathrm{Diff}^+(S^2)$ acts transitively on $\mathcal{R}(S^2)$. A little thought shows that stabilizer of g_0 (the standard round metric on S^2) can be identified with the conformal equivalences of (S^2, g_0), i.e. $\mathrm{PSL}(2,\mathbb{C})$ (by a well-known result in complex analysis). Thus by the orbit-stabilizer theorem, we have

$$\left(\mathcal{C}^\infty(S^2) \rtimes \mathrm{Diff}^+(S^2)\right)/\mathrm{PSL}(2,\mathbb{C}) \cong \mathcal{R}(S^2).$$

Remark 3.2 *Note that Diff*$^+(S^2)$ *preserves* $\mathcal{R}^+(S^2)$ *(or* $\mathcal{R}^-(S^2)$*) since pulling back a metric under a diffeomorphism just "shuffles the curvature around".*

Remark 3.3 *This shows that Diff*$^+(S^2)/PSL(2,\mathbb{C})$ *is contractible, since* $\mathcal{C}^\infty(S^2)$ *and* $\mathcal{R}(S^2)$ *are.*

Now we will need to understand how scalar curvature changes under conformal transformations $g \rightsquigarrow u^{\frac{4}{n-2}}g$, where $u \in \mathcal{C}^\infty(M)$ and the exponent $\frac{4}{n-2}$ is only to simplify later formulas. The formula for $n \geq 3$ is:

$$R_{u^{\frac{4}{n-2}}g}(M) = u^{-\frac{n+2}{n-2}}\left(-\frac{4(n-1)}{n-2}\Delta_g u + R_g(M)u\right) \tag{1}$$

Recall that $\Delta_g u = \mathrm{div}_g \nabla_g u$ is the Laplace-Beltrami operator, where $\mathrm{div}_g X$ for a vector field X is defined by

$$d(\iota_X dV_g) = \mathrm{div}_g X dV_g.$$

In our case, for $n = 2$ and $g = g_0$, we have a simpler formula:

$$\begin{aligned} R_{e^u g_0} &= -e^{-u}\Delta_{g_0}(u) + R_{g_0}(S^2)e^{-u} \\ &= -e^{-u}\Delta_{g_0}(u) + 2e^{-u}. \end{aligned}$$

Let $\mathcal{C}_p^\infty(S^2)$ denote the space of functions $u \in \mathcal{C}^\infty(M)$ such that $R_{e^u g_0} > 0$.
Claim: $\mathcal{C}_p^\infty(S^2)$ is star-shaped about the zero function.

Claim Proof: For $R_{e^u g_0} > 0$, we have (for $0 \leq t \leq 1$)

$$\begin{aligned} R_{e^{tu} g_0} &= -e^{tu}\Delta_{g_0}(tu) + 2e^{-tu} \\ &= e^{-tu}(-t\Delta_{g_0}(u) + 2) \\ &= e^{-tu}(t(e^u R_{e^u g_0} - 2) + 2) \\ &= e^{-tu}(te^u R_{e^u g_0} + 2(1-t)) \\ &> 0 \end{aligned}$$

Thus $\mathcal{C}_p^\infty(S^2)$ is contractible. Now we have an identification

$$\left(\mathcal{C}_p^\infty(S^2) \cdot \mathrm{Diff}^+(S^2)\right)/\mathrm{PSL}(2,\mathbb{C}) \cong \mathcal{R}^+(S^2).$$

Then since $\mathcal{C}_p^\infty(S^2)$ and $\mathrm{Diff}^+(S^2)/\mathrm{PSL}(2,\mathbb{C})$ are contractible, so is $\mathcal{R}^+(S^2)$. ∎

4 Main Proof

We now sketch a proof of Theorem 1.3.

Proof (Theorem 1.3) Let $f : S^i \to \mathcal{R}^-(M)$ be given. We goal is to find an extension

$$\begin{array}{ccc} S^i & \xrightarrow{f} & \mathcal{R}^-(M) \ , \\ \downarrow & \overset{F}{\nearrow} & \\ B_6^{i+1} & & \end{array}$$

where it will convenient to use the ball B_6^{i+1} centered at the origin of radius 6.

- **Step 1:**

  Thinking of B_6^{i+1} as $(S^i \times [0,6]) / (S^i \times \{0\})$, we begin by defining F_1

$$\begin{array}{ccc} S^i & \xrightarrow{f} & \mathcal{R}^-(M) \\ \downarrow & & \downarrow \\ B_6^{i+1} & \xrightarrow{F_1} & \mathcal{R}(M) \end{array}$$

  by the formula

$$F_1(x,t) := \begin{cases} (1-t)g_0 + tf(x) & \text{on } (S^i \times [0,1]) / S^i \times \{0\} \\ f(x) & \text{on } S^i \times [1,6] \end{cases}$$

  for g_0 a fixed metric on M.

- **Step 2:**

  First we'll need a controlled way of connect-summing with a sphere of negative scalar curvature. We'll then use this procedure to alter F_1 to F_2 with

$$\begin{array}{ccc} S^i & \xrightarrow{f} & \mathcal{R}^-(M) \ . \\ \downarrow & & \downarrow \\ B_6^{i+1} & \xrightarrow{F_2} & \mathcal{R}^-_{av}(M) \end{array}$$

  Consider the following data:

  - g_M a fixed metric on M with injectivity radius $\mathrm{inj}(M, g_M) > 5$
  - $\tilde{g}_{S^n}$ any metric on S^n with $\mathrm{inj}(S^n, \tilde{g}_{S^n}) > 5$
  - g any metric on M
  - g_{neg} a fixed negative scalar curvature metric on S^n

– $\Phi_M : (T_pM, g_M) \to (\mathbb{R}^n, g_{euc})$ a fixed linear isometry

– $\Phi_{S^n} : (T_{p'}S^n, g_M) \to (\mathbb{R}^n, g_{euc})$ a fixed linear isometry

Let $f_\lambda : B_5^{\lambda^2 g_M} \setminus \{p\} \to (0,5) \times S^{n-1}$ be defined by

$$f_\lambda(z) := P(\lambda^2(\Phi_M \circ (\exp_p^{\lambda^2 g_M})^{-1}(z)))$$

for

$$P(z) := (|z|, z/|z|).$$

That is, we have a composition

$$(M, \lambda^2 g_M) \xrightarrow{\left(\exp_p^{\lambda^2 g_M}\right)^{-1}} (T_pM, \lambda^2 g_M(p)) \xrightarrow{\Phi_M} (\mathbb{R}^n, g_{euc}) \xrightarrow{P} ((0,5) \times S^{n-1}, g_{\mathbb{R}} \times g_{S^{n-1}})$$

Let $h \in \mathcal{C}^\infty(\mathbb{R})$ be a function with $h \equiv 1$ on $(-\infty, 3]$ and $h \equiv 0$ on $[4, \infty)$. Let $G_\lambda := f_\lambda^*(g_{\mathbb{R}} + g_{S^{n-1}})$. Now define a metric $g_1(\lambda, g)$ on M by

$$g_1(\lambda, g) := h(d_{\lambda^2 g_M}(p, \mathrm{id}_M))G_\lambda + (1 - h(d_{\lambda^2 g_M}(p, \mathrm{id}_M)))\lambda^2 g.$$

Define a metric $g_2(\mu, g_{neg})$ on S^n similarly, using $\tilde{g}_{S^n}$ and Φ_{S^n} instead of g_M and Φ_M. Since we've standardized the metrics in balls, we can perform the connect-sum

$$(M, g_1(\lambda, g))\#(S^n, g_2(\mu, g_{neg})) \rightsquigarrow (M\#S^n, g_\#(\lambda, \mu, g)).$$

Using a continuous family of diffeomorphisms

$$F(\lambda, \mu) : M \to M\#S^n$$

with $F(\lambda, \mu) \equiv \mathrm{id}$ on $M \setminus B_5^{\lambda^2 g_M}(p)$, we get metrics $G(g, \lambda, \mu)$ on M defined by

$$G(g, \lambda, \mu) := F(\lambda, \mu)^* g_\#(\lambda, \mu, g).$$

We now define

$$F_2(\lambda_0, \mu_0, \mu_1, x, t) := \begin{cases} f(x) & \text{on } S^i \times [4,6] \\ ((4-t)\lambda_0^2 + (1 - [4-t]))\, f(x) & \text{on } S^i \times [3,4] \\ (3-t)G(f(x), \lambda_0, \mu_0) + (1 - [3-t])\lambda_0^2 f(x) & \text{on } S^i \times [2,3] \\ G(f(x), \lambda_0, (2-t)\mu_1 + (1-(2-t))\mu_0) & \text{on } S^i \times [1,2] \\ G(F_1(x,t), \lambda_0, \mu_1) & \text{on } \left(S^i \times [0,1]\right)/S^i \times \{0\} \end{cases}$$

Proposition 4.1 *There exist λ_0, μ_0, μ_1 such that $F_2(x,t) := F_2(\lambda_0, \mu_0, \mu_1, x, t)$ is a continuous extension of f with*

$$R_{av}(M, F_2(x,t)) < 0$$

for all x, t.

- **Step 3:** Now we would like a (continuous) way of going from $\mathcal{R}_{av}^-(M)$ to $\mathcal{R}^-(M)$. Specifically, we'll present a mechanism

$$\mathcal{R}_{av}^-(M) \to \mathcal{R}^-(M)$$
$$g \mapsto v(g)^{\frac{4}{n-2}} g$$

(a conformal change). Recall formula (1) for conformal changes:

$$R_{u^{\frac{4}{n-2}} g}(M) = u^{-\frac{n+2}{n-2}} L_g u,$$

where

$$L_g u := -\frac{4(n-1)}{n-2} \Delta_g u + R_g(M) u$$

is the "conformal Laplacian".

We will define $v(g)$ as follows. We define the "Rayleigh quotient" by

$$\begin{aligned} \lambda_1(g) &:= \inf_{u \in \mathcal{C}^\infty(M), |u|_{L^2(M,g)}=1} \int_M \left(\frac{4(n-1)}{n-2} |\nabla_g u|^2 + R_g u^2 \right) dV_g \\ &= \inf_{u \in \mathcal{C}^\infty(M), |u|_{L^2(M,g)}=1} \langle u, L_g u \rangle_{L^2(M,g)} \end{aligned}$$

Proposition 4.2 *There is a unique $v(g) \in \mathcal{C}^\infty(M)$ with*

- $L_g v(g) = \lambda_1(g) v(g)$
- $v(g) > 0$ *and* $\max v(g) = 1$

Remark 4.3 *Setting $u \equiv 1$, we get*

$$\int_M \left(\frac{4(n-1)}{n-2} |\nabla_g u|^2 + R_g u^2 \right) dV_g = \int_M R_g dV_g,$$

so $g \in \mathcal{R}_{av}^-(M)$ implies that $\lambda_1(g) < 0$.

Then for $v(g)^{\frac{4}{n-2}}g$, we have

$$\begin{aligned} R_{v(g)^{\frac{4}{n-2}}g} &= v(g)^{-\frac{n+2}{n-2}} L_g v(g) \\ &= v(g)^{-\frac{n+2}{n-2}} \lambda_1(g) v(g) \\ &< 0, \end{aligned}$$

and hence $v(g)^{\frac{4}{n-2}}g \in \mathcal{R}^-(M)$ as desired.

Finally, set

$$F(x,t) = \begin{cases} f(x) & \text{on } S^i \times [5,6] \\ [(5-t)v(f(x)) + (1-(5-t))]^{\frac{4}{n-2}} f(x) & \text{on } S^i \times [4,5] \\ v(F_2(x,t))^{\frac{4}{n-2}} F_2(x,t) & \text{on } \left(S^i \times [0,4]\right)/S^i \times \{0\} \end{cases}$$

Using (1), one can easily prove

Proposition 4.4 *F is a continuous extension of f with image in $\mathcal{R}^-(M)$.*

∎

Remark 4.5 *Using similar arguments, one can show that the space $\mathcal{R}^{-1}(M)$ of metrics on M with constant scalar curvature -1 is also contractible.*

References

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