Lagrangian Floer Cohomology of $\mathbb{R}P^n \subset \mathbb{C}P^n$

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1 Introduction

In [Oh1], Oh defines Lagrangian Floer cohomology for any monotone Lagrangian $L \subset P$ in a symplectic manifold (P, ω), under a certain topological assumption. More precisely, for any Hamiltonian isotopy $\phi = \{\phi_t\}_{0 \leq t \leq 1}$ of P such that $\phi_1(L)$ intersects L transversally, he defines a relatively graded $\mathbb{Z}_2$-module $I^*(L, \phi : P)$ which is shown to be independent of the isotopy ϕ. This extends the definition of Floer's celebrated homology to many cases with nontrivial $\pi_2(P, L)$.

More precisely, recall that for any Lagrangian $L \subset P$ in a symplectic manifold (P, ω) we have two homomorphisms

$$I_\omega : \pi_2(P, L) \to \mathbb{R}$$
$$I_{\mu,L} : \pi_2(P, L) \to \mathbb{Z}.$$

If $f : (D^2, \partial D^2) \to (P, L)$ is a smooth map of pairs, $I_\omega([f])$ is defined by

$$I_\omega([f]) = \int_{D^2} f^*\omega.$$

To define $I_{\mu,L}$, we first pick a symplectic trivialization of f^*TP and use this to identify $f|\partial D^2$ with a map $f_\partial : \partial D^2 \to \Lambda(\mathbb{C}^n)$, where $\Lambda(\mathbb{C}^n)$ is the space of Langrangian linear subspaces of $\mathbb{C}^n$. Letting $\mu \in H^1(\Lambda(\mathbb{C}^n), \mathbb{Z})$ denote the Maslov cycle, we define $I_{\mu,L}$ by

$$I_{\mu,L}([f]) = \mu([f_\partial]).$$

We say that $L \subset P$ is *monotone* if

$$I_{\mu,L} = \lambda I_\omega$$

for some $\lambda > 0$. Let Σ_L denote the positive generator of the subgroup $I_{\mu,L}(\pi_2(P,L)) \subset \mathbb{Z}$. We define

$$I(L, \phi : P) = \{x \in \phi_1(L) \cap L \mid [t \mapsto \phi_t^{-1}(x)] = 0 \in \pi_1(P, L)\}$$

and

$$\mathscr{C}^* = \mathbb{Z}_2\langle I(L, \phi : P)\rangle.$$

The relevant theorem from [Oh1] can now be stated as follows.

Theorem 1.1 *Let L be a monotone Lagrangian submanifold in (P, ω) and $\phi = \{\phi_t\}_{0\le t\le 1}$ be a Hamiltonian isotopy of P such that L intersects $\phi_1(L)$ transversally. Suppose $\Sigma_L \geq 3$. Then there exists a homomorphism*

$$\delta : \mathscr{C}^* \to \mathscr{C}^*$$

with $\delta \circ \delta = 0$ such that the quotients

$$I^*(L, \phi : P) := Ker\, \delta / Im\, \delta$$

are isomorphic as relatively $\mathbb{Z}/\Sigma$-graded $\mathbb{Z}/2$ modules for any Hamiltonian isoptopy ϕ, provided L intersects $\phi_1(L)$ transversally.

We denote the common module by $I^*(L : P)$.

Remark 1.2 *It is important to note here that, when well-defined, $I^*(L, \phi : P)$ ultimately a Hamiltonian isotopy invariant of the Lagrangian L sitting inside the symplectic manifold (P, ω), even though its construction a priori depends on the isotopy ϕ and a choice of appropriate almost complex structure J on P.*

Now let us consider the Lagrangian $\mathbb{R}P^n \subset \mathbb{C}P^n$, where $\mathbb{C}P^n$ is given the standard Kahler symplectic form ω coming from the Fubini-Study metric and $\mathbb{R}P^n$ is the fixed point set of the anti-holomorphic involutive isometry σ given in homogeneous coordinates by

$$\sigma([z_0 : z_1 : ... : z_n]) = [\overline{z_0} : \overline{z_1} : ... : \overline{z_n}].$$

As we show in Section 3, $\mathbb{R}P^n$ here is actually a monotone Lagrangian, and we indeed have $\Sigma_{\mathbb{R}P^n} = n + 1 \geq 3$ for $n \geq 2$. Thus $I^*(\mathbb{R}P^n : \mathbb{C}P^n)$ is well-defined for $n \geq 2$ [1] and our goal in these notes is to prove the following theorem, following [Oh2].

[1] Actually $I^*(\mathbb{R}P^1 : \mathbb{C}P^1)$ is also well-defined and gives the expected outcome, as Oh shows by a more careful analysis of the disk bubbling that can occur.

Theorem 1.3 *Assume $n \geq 2$, and let $\mathbb{R}P^n$ and $(\mathbb{C}P^n, \omega)$ be as above. Then*

$$I^*(\mathbb{R}P^n : \mathbb{C}P^n) \cong H^*(\mathbb{R}P^n, \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{n+1}$$

as relatively $\mathbb{Z}/(n+1)$-graded modules.

One immediate corollary is a version of the Arnold conjecture:

Corollary 1.4 *For any Hamiltonian isotopy $\phi = \{\phi_t\}_{0 \leq t \leq 1}$ of $\mathbb{C}P^n$ such that $\mathbb{R}P^n$ intersects $\phi_1(\mathbb{R}P^n)$ transversally, we have*

$$\#(\mathbb{R}P^n \cap \phi_1(\mathbb{R}P^n)) \geq n+1 = dim_{\mathbb{Z}/2} H^*(\mathbb{R}P^n, \mathbb{Z}/2).$$

In order to prove Theorem 1.3, we should first give some more details about the definition of $\delta : \mathscr{C}^* \to \mathscr{C}^*$. Roughly speaking, δ counts holomorphic strips with Lagrangian boundary conditions between intersection points in $L \cap \phi_1(L)$. For this we pick an almost complex structure J on P which is compatible with ω (i.e. $\omega(\cdot, J\cdot)$ defines a Riemannian metric on P). It can be shown that for "generic" such J, the relevant moduli spaces of holomorphic strips form manifolds, whose dimensions are controlled by the so-called "Maslov-Viterbo index". For sufficiently generic J, Oh uses a version of Gromov's Compactness Theorem and the assumption $\Sigma_L \geq 3$ to show that δ is well-defined (i.e. the relevant count is finite) and $\delta \circ \delta = 0$.

To make this rigorous, we need to make some definitions. Let $x, y \in I(L, \phi : P)$.

Definition 1.5

1. $\Theta := \{a + bi \in \mathbb{C} \mid 0 \leq b \leq 1\}$
2. $\Omega_\phi := \{z : I \to P \mid z(0) \in L,\ z(1) \in \phi_1(L), [t \mapsto \phi_t^{-1} z(t)] = 0 \in \pi_1(P, L)\}$
3. $\mathscr{P}_\phi := \{u \in L^2_k(\Theta, P) \mid u(\tau, 0) \subset L,\ u(\tau, 1) \subset \phi_1(L),\ u(\tau, \cdot) \in \Omega_\phi \ \forall\ \tau\}$
4. $\mathcal{M}_{J,\phi} := \{u \in \mathscr{P}_\phi \mid \overline{\partial}_J u := \frac{\partial u}{\partial \tau} + J \frac{\partial u}{\partial t} = 0,\ \int_\Theta \left|\frac{\partial u}{\partial \tau}\right|^2 dt d\tau < \infty\}$
5. $\mathcal{M}_{J,\phi}(x, y) := \{u \in \mathcal{M}_{J,\phi} \mid \lim_{\tau \to \infty} u = x,\ \lim_{\tau \to -\infty} u = y\}$
6. $\widehat{\mathcal{M}}_{J,\phi}(x, y) := \mathcal{M}_{J,\phi}(x, y)/\mathbb{R}$.
7. $\mathscr{L}_u := \{\xi \in L^2_{k-1}(\Theta, TP) \mid \xi(\theta) \in T_{u(\theta)} P\}$.

Here $\mathscr{L}$ forms a Banach bundle over $\mathscr{P}_\phi$, and $\overline{\partial}_J$ gives a section $\mathscr{P}_\phi \to \mathscr{L}$. We denote by

$$E_u = D\overline{\partial}_J(u) : T_u \mathscr{P}_\phi \to \mathscr{L}_u$$

the covariant linearization of $\overline{\partial}_J$ at u.

Now we are ready to state under what conditions we can define $I^*(L, \phi : P) = \text{Ker}\delta/\text{Im}\delta$. In fact, the complex $\mathscr{C}^*$ will depend on the choice of a "nice" compatible almost complex structure J, although it can be shown that the cohomology $I^*(L,\phi : P)$ is independent of the choice of J. Indeed, for a given J, the chain complex $\delta : \mathscr{C}^* \to \mathscr{C}^*$ can be defined by

$$\delta x := \sum_{y \in I(L,\phi:P)} y\langle y, \delta x\rangle,$$
$$\langle y, \delta x\rangle := \sum_{y \in I(L,\phi,P)} \# \left(\widehat{\mathcal{M}}_{J,\phi}(y,x)\right) \mod 2,$$

where $\# \left(\widehat{\mathcal{M}}_{J,\phi}(x,y)\right)$ denotes the number of zero-dimensional components of $\widehat{\mathcal{M}}_{J,\phi}(x,y)$, under the conditions:

1. (ϕ, J) is *regular*, i.e. $\mathrm{Coker} E_u = 0$ for all $u \in \mathcal{M}_{J,\phi}(x,y)$ and for all $x, y \in I(L,\phi : P)$
2. $\# \left(\widehat{\mathcal{M}}_{J,\phi}(x,y)\right)$ is finite for all $x, y \in I(L,\phi : P)$
3. $\sum_{y \in I(L,\phi)} \langle x, \delta y\rangle \langle y, \delta z\rangle = 0 \in \mathbb{Z}/2$ for any $x, z \in I(L,\phi : P)$

We future ease, we'll call a pair (ϕ, J) satisfying these conditions *admissible*. Evidently any admissible pair (ϕ, J) gives rise to a chain complex $(\mathscr{C}^*, \delta)$ with cohomology equal to $I^*(L : P)$.

We can now break up the proof of Theorem 1.3 as follows. In Section 2 we show how to pick a convenient Hamiltonian isotopy ϕ satisfying $|I(L,\phi : P)| = |L \cap \phi_1(L)| = n+1$ by exploiting the automorphism group of $\mathbb{C}P^n$. We then show that the standard integrable J on $\mathbb{C}P^n$ indeed satisfies the above conditions, using:

Proposition 1.6 *(Regularity) Let $L = \mathbb{R}P^n \subset \mathbb{C}P^n$ be the standard one and (J,ϕ) as above. Then the pair (ϕ, J) is regular, i.e. the linearization E_u is surjective for all $u \in \mathcal{M}_{J,\phi}$.*

Proposition 1.7 *(Compactness) Under the above hypotheses, the zero-dimensional component of $\widehat{\mathcal{M}}_{J,\phi}$ is compact and the one-dimensional component of $\widehat{\mathcal{M}}_{J,\phi}$ is compact up to the splitting of two-trajectories.*

Proposition 1.7 implies that $\delta \circ \delta = 0$ in the usual way by noticing that compact one-dimensional manifolds have an even number of boundary points and using the standard gluing technique for broken trajectories.

Finally, we show:

Proposition 1.8 *(Vanishing) Under the same hypotheses, $\delta \equiv 0$.*

In summary, given the construction of Langrangian Floer Cohomology for monotone Lagrangians with $\Sigma \geq 3$ as stated in Theorem 1.1, the computation of $I^*(\mathbb{R}P^n : \mathbb{C}P^n)$ involves the following steps:

- Show that $\mathbb{R}P^n \subset \mathbb{C}P^n$ is a monotone Lagrangian with $\Sigma \geq 3$

- Choose a convenient Hamiltonian isotopy ϕ and show that it has satisfies $|I(L, \phi : P)| = |L \cap \phi_1(L)| = n + 1$
- Choose an almost complex structure J, namely the standard integrable one, and show that the pair (ϕ, J) is admissible, i.e. (ϕ, J) is regular and we have appropriate compactness statements to conclude that δ is well-defined and $\delta \circ \delta = 0$
- Show that the boundary operator δ is trivial.

In the following sections, we give a rough sketch of these steps. We refer the reader to [Oh2] for more of the details.

2 Choosing a Convenient Isotopy ϕ

Recall that $G = PU(n+1)$ is the group of biholomorphic isometries of $\mathbb{C}P^n$, and has a maximal torus $T^n \subset G$. Actually the action of G is Hamiltonian, with moment map of the action of T^n, $\Phi : \mathbb{C}P^n \to \mathfrak{t}^*$, given by

$$f_\xi(x) := \langle \Phi(x), \xi \rangle = \frac{\overline{x}^t \xi x}{2\pi i ||x||^2}$$

where $x = (x_0, x_1, ..., x_n)$, $||x||^2 = x_0 \overline{x_0} + ... + x_n \overline{x_n}$ and $\xi \in \mathfrak{t} =$ the Lie algebra of T^n. Using this one easily checks that

$$\sigma^* f_\xi = f_\xi,$$

where σ is the anti-holomorphic involutive isometry as before. From this we have

$$\sigma^* \xi_{\mathbb{C}P^n} = -\xi_{\mathbb{C}P^n},$$

where $\xi_{\mathbb{C}P^n}$ is the vector field on $\mathbb{C}P^n$ associated to ξ by the action of T^n. Letting ψ_t denote the flow of $\xi_{\mathbb{C}P^n}$, we then have

$$\sigma \psi_t \sigma = \psi_t^{-1}.$$

Now since $\xi_{\mathbb{C}P^n}$ is orthogonal to $\mathbb{R}P^n$, we have

$$\mathbb{R}P^n \cap \psi_t(\mathbb{R}P^n) = \text{Crit}(f_\xi)$$

for $t \neq 0$ sufficiently small. One can check that

$$\#(\text{Crit} f_\xi) = n + 1.$$

We now choose $\xi \in \mathfrak{t}$ such the corresponding flow ψ_t is periodic with period one, and then define $\phi_t = \psi_{t/2^N}$ for N sufficiently large. This gives an flow ϕ_t such that

- $\phi_1^{2^N} = \mathrm{id}$
- $\#(L \cap \phi_t(L)) = n+1$
- $\sigma\phi_t\sigma = \phi_t^{-1}$
- ϕ_t is a biholomorphic isometry for all t.

Remark 2.1 *We note that $\pi_1(\mathbb{C}P^n, \mathbb{R}P^n) = 0$, and therefore we need worry about whether paths created in $\pi_1(P, L)$ are trivial.*

3 Monotonicity and $\Sigma \geq 3$

In this section we prove that the standard $\mathbb{R}P^n \subset \mathbb{C}P^n$ is a monotone Lagrangian. Firstly, we claim that $P = \mathbb{C}P^n$ is a monotone symplectic manifold, i.e. there exists some $\lambda > 0$ such that for any $u : S^2 \to P$ we have

$$c_1(u^* T\mathbb{C}P^n)[S^2] = \alpha \int_{S^2} u^*\omega.$$

Indeed, $\pi_2(P) \cong \mathbb{Z}$ is generated by $\mathbb{C}P^1 \subset \mathbb{C}P^n$, i.e. a J-holomorphic map

$$u : S^2 \to \mathbb{C}P^n,$$

which therefore has $\int_{S^2} u^*\omega$ equal to the symplectic area of u, which is positive. On the other hand, recall that we have the characterization

$$T\mathbb{C}P^n = \mathrm{Hom}_{\mathbb{C}}(\gamma, \gamma^{\perp}),$$

where γ is the tautological line bundle over $\mathbb{C}P^n$. Then writing 1 for the trivial line bundle, we have

$$\begin{aligned} T\mathbb{C}P^n \oplus 1 &\cong \mathrm{Hom}_{\mathbb{C}}(\gamma, \gamma^{\perp}) \oplus \mathrm{Hom}_{\mathbb{C}}(\gamma, \gamma) \\ &\cong \mathrm{Hom}_{\mathbb{C}}(\gamma, \oplus^{n+1} 1) \\ &\cong \overline{\gamma}^{n+1}. \end{aligned}$$

Therefore

$$c_1(T\mathbb{C}P^n) = (n+1)c_1(\overline{\gamma}),$$

and it follows by naturality of Chern classes that

$$c_1(u^* T\mathbb{C}P^n)[S^2] = n+1,$$

which is also positive.

Before proving that $\mathbb{R}P^n$ is monotone, we record a useful lemma.

Lemma 3.1 *Let $f, f' : (D^2, \partial D^2) \to (P, L)$ be smooth maps of pairs with*

$$f|_{\partial D^2} = f'|_{\partial D^2}.$$

Let u denote the corresponding map from $S^2 = D^2 \cup \overline{D}^2$ to P defined by gluing, i.e.

$$u(z) = \begin{cases} f(z) & : z \in D^2 \\ f'(z) & : z \in \overline{D}^2. \end{cases}$$

Then we have

$$\mu(f) - \mu(f') = 2c_1(P, \omega)[u].$$

Proof Indeed, since any symplectic vector bundle over D^2 is trivial, we can view $u^*(P, \omega)$ as being defined by an element $[u_\partial] \in \pi_1(\mathrm{Sp}(2n)) \cong \mathbb{Z}$. Then $[u_\partial] \in \mathbb{Z}$ gives $c_1(P, \omega)[u]$, and its image under the map

$$\pi_1(\mathrm{Sp}(2n)) \to \pi_1(\Lambda(\mathbb{C}^n))$$

gives $\mu(f) - \mu(f')$. But the above map can be identified with

$$\times 2 : \mathbb{Z} \to \mathbb{Z}.$$

Now we establish monotonicity using the following lemma:

Lemma 3.2 *Let (P, ω) be a monotone symplectic manifold with monotonicity constant $\alpha > 0$, and let $\sigma : P \to P$ be an anti-symplectic involution with nonempty fixed point set $L = Fix\ \sigma$. Then L is a monotone Lagrangian.*

Proof Let $f : (D^2, \partial D^2) \to (P, L)$ be a smooth map of pairs, and let $f'(z) = \sigma \circ f(z)$. Then $f|_{\partial D^2} = f'|_{\partial D^2}$ and so we can glue f and f' as in Lemma 3.1 to get a map $u : S^2 \to P$. From Lemma 3.1, we have

$$2\mu(f) = \mu(f) - \mu(f') = 2c_1(u),$$

i.e.

$$\mu(f) = c_1(u).$$

An easy calculation also shows that we have

$$\int_{S^2} u^*\omega = 2 \int_{D^2} f^*\omega$$

since σ is anti-symplectic. Thus we have

$$\mu(f) = c_1(u) = \alpha[\omega](u) = 2\alpha[\omega](f),$$

i.e.

$$I_{\mu,L}([f]) = 2\alpha I_\omega([f]).$$

Remark 3.3 *Note that from the formula $\mu(f) = c_1(u)$ and our above computation it easily follows that $\Sigma_{\mathbb{R}P^n} = n + 1$.*

4 Compactness

In this section our goal is to prove Proposition 1.7, assuming regularity of (ϕ, J). This will follow from a form of Gromov's Compactness Theorem. Roughly speaking, this says that for any sequence $u_i \in \mathcal{M}_{J,\phi}(x,y)$ with constant Maslov-Viterbo index I and uniformly bounded energy, there exists a subequence converging to some $(\underline{u}, \underline{v}, \underline{w})$, where $\underline{u}$ is a broken k-trajectory in $\widehat{\mathcal{M}}_{J,\phi}$, $\underline{v}$ is a collection of finite energy J-holomorphic spheres, and $\underline{w}$ is a collection of finite energy J-holomorphic disks. Moreover, we have the following index formula:

$$I = \sum_{i=1}^{k} \mathrm{Index}(u_i) + 2\sum_j c_1(v_j) + \sum_l \mu(w_l).$$

Here $\mathrm{Index}(u_i)$ denotes the Maslov-Viterbo index of u_i, which can also be shown to be the local dimension of $\mathcal{M}_{J,\phi}$ near u_i, and therefore in particular is nonnegative. Moreover, since the v_j's and w_l's are J-holomorphic, monotonicity implies that the second and third sums above must also be nonnegative. But for nontrivial v_j or w_l we would then have

$$|2c_1(v_j)|, |\mu(w_l)| \geq \Sigma \geq 3.$$

This shows that for $I = 1, 2$ there can be no sphere or disk bubbles, hence the Proposition.

5 Triviality of the Boundary Operator

Next we prove that $\delta \equiv 0$, again assuming regularity of (ϕ, J). By the definition of δ, it suffices to show that the finite number $\#\left(\widehat{\mathcal{M}_{J,\phi}}(x,y)\right)$ is always even. We will exhibit a fixed point free involution on $\widehat{\mathcal{M}_{J,\phi}}(x,y)$ which associates $u \in \widehat{\mathcal{M}_{J,\phi}}(x,y)$ with

$$\overline{u} = \phi_1^{2^l} u$$

for some $1 < l \leq N-1$ (recall that $\phi_1^{2^N} = \mathrm{id}$).

Using the relation $\sigma\phi_1\sigma = \phi_1^{-1}$, we have for any $p \in L = \mathrm{Fix}\,\sigma = \mathbb{R}P^n$:

$$\sigma\phi_1^{2^{N-1}}(p) = \sigma\phi_1^{2^{N-1}}\sigma(p) = \left(\phi_1^{2^{N-1}}\right)^{-1}(p) = \phi_1^{2^{N-1}}(p),$$

hence $\phi_1^{2^{N-1}}(p) \in L$. Then since $\phi_1^{2^{N-1}}(\mathbb{R}P^n) = \mathbb{R}P^n$ and $\phi_1^{2^{N-1}}(\phi_1(\mathbb{R}P^n)) = \phi_1(\mathbb{R}P^n)$, it follows that for $u \in \mathcal{M}_{J,\phi}(x,y)$ we have again $\phi_1^{2^{N-1}}(u) \in \mathcal{M}_{J,\phi}(x,y)$.

Now if $\phi_1^{2^{N-1}}(u) \not\equiv u$, we set $\overline{u} := \phi_1^{2^{N-1}}(u)$ (one can show that $\overline{u}$ cannot be a translation of u since ϕ_t is perpendicular to L). On the other hand, if $\phi_1^{2^{N-1}}(u) \equiv u$, we can repeat the above, with N replaced by $N-1$, to get an element $\phi_1^{2^{N-2}}(u) \in \mathcal{M}_{J,\phi}(x,y)$. As before, if $\phi_1^{2^{N-2}}(u) \not\equiv u$, we set $\overline{u} := \phi_1^{2^{N-2}}(u)$, otherwise we repeat the process. By choosing N sufficiently large from the beginning so that

$$\phi_1^2 u \not\equiv u$$

for any such u, we can guarantee that this process eventually terminates. Moreover, it is easy to check that the pairing $u \mapsto \overline{u}$ indeed gives a well-defined fixed point free involution on $\widehat{\mathcal{M}_{J,\phi}}(x,y)$.

6 Regularity of (ϕ, J)

Finally, we sketch a proof of Proposition 1.6, which we have been postponing until now. Let $u \in \mathcal{M}_{J,\phi}(x,y)$. Recall that $\overline{\partial_J}$ gives a section of the bundle

$$\mathscr{L} \to \mathscr{P}_\phi,$$

where

$$\begin{aligned} \mathscr{L}_u &= \{\xi \in L^2_{k-1}(\Theta, TP) \mid \xi(\theta) \in T_{u(\theta)}P\} \\ \mathscr{P}_\phi &= \{u \in L^2_k(\Theta, P) \mid u(\tau,0) \subset L,\ u(\tau,1) \subset \phi_1(L)\ \forall\, \tau\} \end{aligned}$$

Our goal is to show that the covariant linearization $E_u = D\overline{\partial}_J(u) : T_u\mathscr{P}_\phi \to \mathscr{L}_u$ at u is surjective.

Let $\xi \in T_u\mathscr{P}_\phi$, and let u_s be a path in $\mathscr{P}_\phi$ with $u_0 = u$ and $(d/ds)|_{s=0}u_s = \xi$. Then we have

$$E_u(\xi) = \nabla_s|_{s=0}\overline{\partial}_J(u_s).$$

Using the fact that $\nabla J = 0$, a short computation shows that

$$E_u(\xi) = (\nabla_\tau + J\nabla_t)\xi,$$

i.e. E_u looks like a covariant version of the $\overline{\partial}_J$ operator.

Let $(E_u)^*$ be the adjoint of E_u. Then to show that $\mathrm{Coker}(E_u) = 0$, it will suffice to show that $\eta = \mathrm{Ker}(E_u)^*$ implies that $\eta = 0$. Using the relation

$$\langle \xi, (E_u)^*\eta\rangle_2 = \langle E_u\xi, \eta\rangle_2$$

and massaging the right hand side, one can prove the following characterization of the cokernel:

$$\mathrm{Coker}E_u = \{\eta \in L^2_{k-1}(\Theta, u^*TP) \mid -\nabla_\tau\eta + J\nabla_t\eta = 0,\ \eta(\tau,1) \in T\phi_1(\mathbb{R}P^n),\ \eta(\tau,0) \in T\mathbb{R}P^n\}.$$

Now we show how by reflecting u $2^N - 1$ times we can get a (finite energy) J-holomorphic map from the cylinder

$$C_{2^N} = (\mathbb{R} \times i[0,2^N])/((a,0) \sim (a,2^N))$$

to P. Indeed, let

$$\begin{aligned} \sigma_1 &:= \phi_1\sigma\phi_1^{-1}, \\ u_1(\tau,t) &:= \sigma_1 u(\tau, 1-t). \end{aligned}$$

Note that since σ is anti-holomorphic, u_1 is J-holomorphic, and one can show that

$$\text{Fix } \phi_1\sigma\phi_1^{-1} = \phi_1(\mathbb{R}P^n)$$
$$u_1(\tau, 0) \in \phi_1(L)$$
$$u_1(\tau, 1) \in \phi_1^2(u(\tau, 0)).$$

Similarly, let

$$\sigma_2 := \phi_1^2\sigma\phi_1^{-2}$$
$$u_2(\tau, t) := \sigma_2 u_1(\tau, 1 - t).$$

We can repeat this process, defining $u_3, u_4, ...$, and it is not hard to show using $\phi_1^{2^N} = \text{id}$ that $u_{2^N} \equiv u$, and we therefore get the promised J-holomorphic map

$$C_{2^N} \to P.$$

We can now appeal to a standard removal of singularities theorem:

Theorem 6.1 *(Removal of singularities) Let (P, w) be a symplectic manifold with compatible almost complex structure J, and let $u : D^2 \setminus \{0\} \to P$ be a J-holomorphic map such that $\int_{D^2\setminus\{0\}} u^*\omega < \infty$. Then u extends to a J-holomorphic map on D^2.*

By the removal of singularities theorem and the fact that C_{2^N} is conformally equivalent to $\mathbb{C}P^1 \setminus \{0, \infty\}$, we can extend our map $C_{2^N} \to P$ to a J-holomorphic map $f : \mathbb{C}P^1 \to \mathbb{C}P^n$.

Now by applying the same reflection process to our η (which we are trying to show is identically 0), we get a section $\overline{\eta}$ of $f^*(T\mathbb{C}P^n)$ which is anti-holomorphic, which corresponds to a holomorphic section of $f^*(T^*\mathbb{C}P^n)$. Now the fact that $\eta = 0$ follows from the following classical result:

Lemma 6.2 *Let $f : \mathbb{C}P^1 \to \mathbb{C}P^n$ be a non-constant holomorphic map with respect to the standard integrable almost complex structures on $\mathbb{C}P^1$ and $\mathbb{C}P^n$. Then there is no nontrivial holmorphic section of $f^*(T^*\mathbb{C}P^n)$.*

Proof By Grothendieck's splitting theorem for holomorphic vector bundles over $\mathbb{C}P^1$, $E := f^*(T^*\mathbb{C}P^n)$ splits as a direct sum of holomorphic line bundles

$$E = L_1 \oplus ... \oplus L_n.$$

Using the large symmetry group of $\mathbb{C}P^n$, it is not hard to show that each L_i must admit a nontrivial holomorphic section which is zero at a point. This means that $c_1(L_i) > 0$ for each i, and therefore for each i we have

$$c_1(L_i^*) = -c_1(L_i) < 0.$$

Since $f^*(T^*\mathbb{C}P^n) \cong E^* \cong L_1^* \oplus ... \oplus L_n^*$, $f^*(T^*\mathbb{C}P^n)$ cannot admit a nontrivial holomorphic section.

References

[Flo] A. Floer. Morse theory for Lagrangian intersections. *J. Differential Geom* **28**(1988), 513–547.

[Oh1] Y.G. Oh. Floer cohomology of lagrangian intersections and pseudo-holomorphic disks I. *Communications on Pure and Applied Mathematics* **46**(1993), 949–993.

[Oh2] Y.G. Oh. Floer cohomology of lagrangian intersections and pseudo-holomorphic disks II:(CPn, RPn). *Communications on Pure and Applied Mathematics* **46**(1993), 995–1012.