Positive Scalar Curvature and Surgery

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1 Introduction

Here is the main result we wish to discuss:

Theorem 1.1 *(Gromov-Lawson, Schoen-Yau) [GL, RS, SY] Let M be a Riemannian manifold with everywhere positive scalar curvature (p.s.c.), and suppose that M' is obtained from M by performing a surgery of codimension $q \geq 3$. Then M' can also be given a p.s.c. metric.*

By way of review, let us first recall the basic premise of surgery theory. We observe that for S^p and D^p the sphere and disk respectively of dimension p, we have

$$\partial(S^p \times D^q) = S^p \times S^{q-1} = \partial(D^{p+1} \times S^{q-1}).$$

Given an embedded $S^p \times D^q \subset M$, let

$$M' = M \setminus (S^p \times D^q) \cup_{S^p \times S^{q-1}} (D^{p+1} \times S^{q-1}).$$

That is, M' is obtained by removing an embedded $S^p \times D^q$ and replacing it with $D^{p+1} \times S^{q-1}$, glued along the common boundary. We say that M' is obtained from M by a surgery of dimension p (or codimension q).

Example 1.2

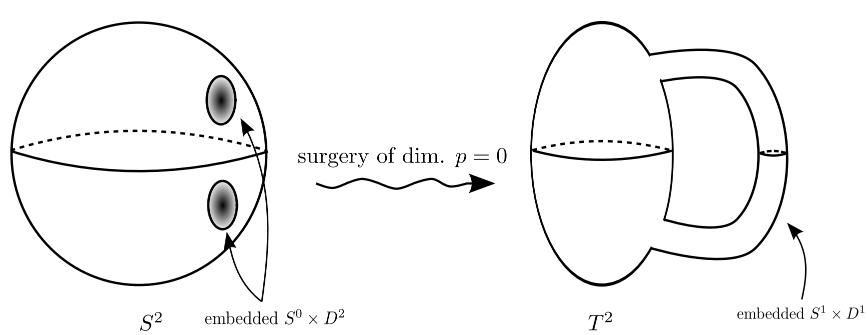


Figure 1: Surgery of dimension 0

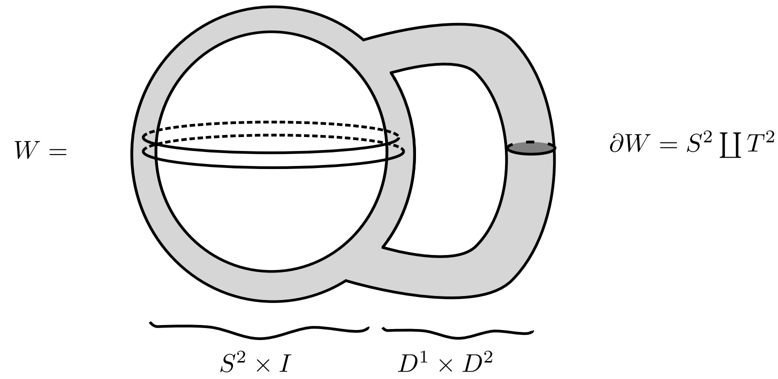


Figure 2: The trace of a surgery of dimension 0

Remark 1.3 *To specify the embedded $S^p \times D^q \subset M$, it suffices to specify an embedded $S^p \subset M$ with a trivial normal bundle, since then a tubular neighborhood does the trick.*

Remark 1.4 *Two compact oriented manifolds are related by a sequence of surgeries (of any dimension) if and only if they are (oriented) corbodant.*

- $\Longleftarrow$*: Follows from standard Morse theory [Mil, MSS]*
- $\Longrightarrow$*: We construct the trace of a p-surgery:*

$$W := M \times I \cup_{S^p \times D^q} (D^{p+1} \times D^q).$$

Then $\partial W = \overline{M} \coprod M'$.

Example 1.5 *See Figure 2.*

Remark 1.6 *Since cobordism classes of manifolds (with various decorations) are in many cases well-understood, this foreshadows the beginning of the classification of (high dimensional) manifolds, leaving us with the task of understanding how a manifold can change under surgeries.*

2 Some Consequences

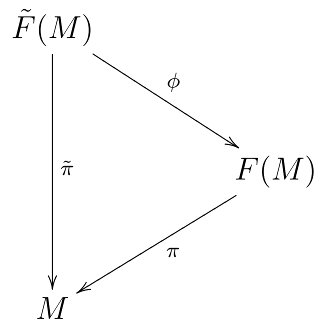
Before discussing the proof of Theorem 1.1, we give some consequences.

Theorem 2.1 *(Gromov-Lawson) [GL] If M is a closed, simply connected manifold of dimension $n \geq 5$, $w_2(M) = 0$ (i.e. M is spin), and if M is spin-cobordant to a manifold admitting p.s.c., then M also admits a metric of p.s.c.*

Here $w_2(M) \in H^2(M; \mathbb{Z}/2)$ denotes the second Stiefel-Whitney class of the tangent bundle of M.

Theorem 2.2 *(Gromov-Lawson) [GL] If M is a closed, simply connected manifold of dimension $n \geq 5$ and if $w_2(M) \neq 0$, then M admits a p.s.c. metric.*

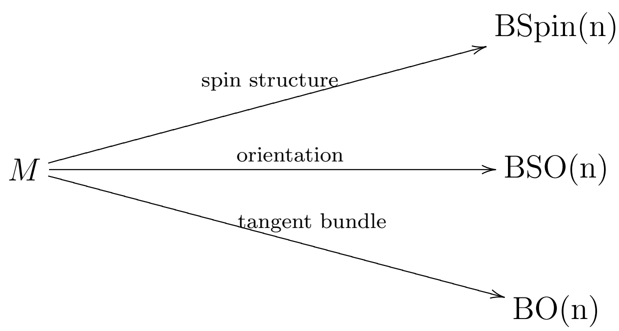
Recall that a spin structure on an oriented manifold M^n is a principle Spin(n)-bundle $\tilde{F}(M)$ over M sitting in a diagram of the form



with ϕ a double cover, $F(M)$ the bundle of orthonormal oriented n-frames on M, π and $\tilde{\pi}$ the obvious projections, and such that ϕ is equivariant with respect to the covering homomorphism

$$\mathrm{Spin}(n) \to \mathrm{SO}(n).$$

The situation can also be viewed using classifying maps as



In fact, the obstructions to finding an orientation and spin structure are precisely $w_1(M) \in H^1(M; \mathbb{Z}/2)$ and $w_2(M) \in H^2(M; \mathbb{Z}/2)$ respectively. Therefore an oriented manifold is spin if and only if $w_2(M) = 0$.

Remark 2.3 *Orientations and spin structures (or more generally so-called "(B, f)-structures" [Wes]) give rise to the cobordism rings Ω_*^{Spin} and Ω_*^{SO} respectively. In these rings we have:*

- ***elements****: (decorated) cobordism classes of (decorated) manifolds*
- ***addition****: disjoint union of manifolds*
- ***multiplication****: cartesian product of manifolds.*

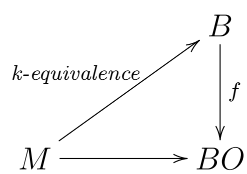
n	Ω_n^{Spin}	Ω_n^{SO}
0	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}/2$	0
2	$\mathbb{Z}/2$	0
3	0	0
4	$\mathbb{Z}$	$\mathbb{Z}$
5	0	$\mathbb{Z}/2$
6	0	0
7	0	0
8	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$

- ***grading***: *dimension of manifolds*

In low dimensions, these are given by [LM]

In general, we have

Theorem 2.4 *(Bordism Theorem) Let M be a closed manifold of dimension $n \geq 5$, with a (B, f)-structure such that the classifying map $M \to B$ is a k-equivalence for $k < \lfloor \frac{n+1}{2} \rfloor$.*



Then if M is (B, f)-cobordant to N, M is obtained from N by a sequence of surgeries of codimension $q > k$.

This puts us in a strong position to apply Theorem 1.1:

Corollary 2.5 *If $M \to B$ is a 2-equivalence (ex: $B = BSpin$ and $M \to BSpin$ is a 2-equivalence) and M is (B, f)-cobordant to a manifold admitting p.s.c., then M admits p.s.c.*

Proof idea for Theorems 2.1, 2.2, and 2.4: Let W be a (SO or Spin) bordism between M and N, with $\pi_1(M) = 0$ and N admitting a p.s.c. metric. Using surgery on W, show that we can assume $\pi_1(N) = \pi_1(W) = 0$ and $H_i(W, M) = 0$ for small i. Following the proof of the h-cobordism theorem [MSS, Lüc], we can get rid of the low dimensional handles of W (with respect to M) to conclude that N is obtained from M by surgeries of dimension ≥ 3, i.e. M is obtained from N by surgeries of codimension ≥ 3. We can then appeal to Theorem 1.1.

To complete the proof of Theorem 2.2, show that (in high dimensions) Ω_*^{SO} is generated by manifolds admitting p.s.c., namely complex projective spaces (which can be given the Fubini study metric) and "friends" of these.

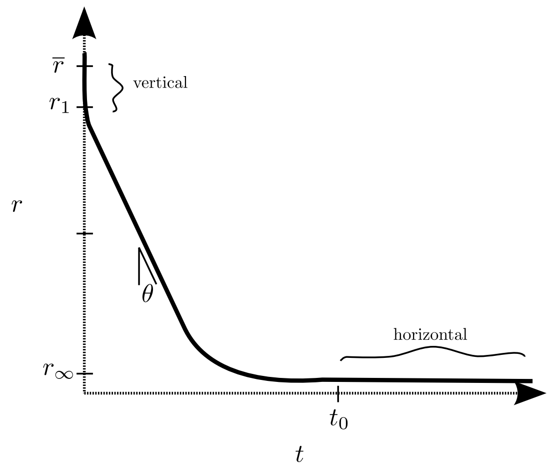


Figure 3: The curve γ

3 The Main Proof

We are now ready to prove Theorem 1.1, after a quick remark about the codimension assumption.

Remark 3.1 *The assumption in Theorem 1.1 that the surgeries must be codimension ≥ 3 cannot be removed. Recall that no torus of any dimension admits positive scalar curvature [SY]. On the other hand, we know for example that $\mathbb{T}^2, \mathbb{T}^3, \mathbb{T}^6$, and $\mathbb{T}^7$ are obtained by surgeries from S^2, S^3, S^6, and S^7 respectively, since $\Omega_2^{SO} = \Omega_3^{SO} = \Omega_6^{SO} = \Omega_7^{SO} = 0$. Evidently some of the surgeries must be of codimension < 3, since the spheres admit positive scalar curvature.*

Proof (Theorem 1.1) For simplicity, we first consider the case of codimension n, i.e. surgery on S^0, i.e. "connected sum".

Let M be an n-dimensional p.s.c. manifold, and let D^n be a small ball of radius $\overline{r}$ in geodesic normal coordinates on M (so that radial lines are geodesics). We'll construct some suitable $\mathcal{C}^\infty$ curve γ in the "t-r" plane, and consider

$$T = \{(x,t) \in D^n \times \mathbb{R} \ : \ (|x| = r, t) \in \gamma\},$$

where γ looks roughly as in Figure 3.

We give T the induced metric from $D^n \times \mathbb{R}$. See Figure 4.

We choose γ to satisfy:

1. γ lies in the region $0 < r \leq \overline{r}$ of the t-r plane

2. γ begins with a vertical line segment $t = 0$, $r_1 \leq r < \overline{r}$

3. γ ends with a horizontal line segment $r = r_\infty > 0$ for r_∞ very small

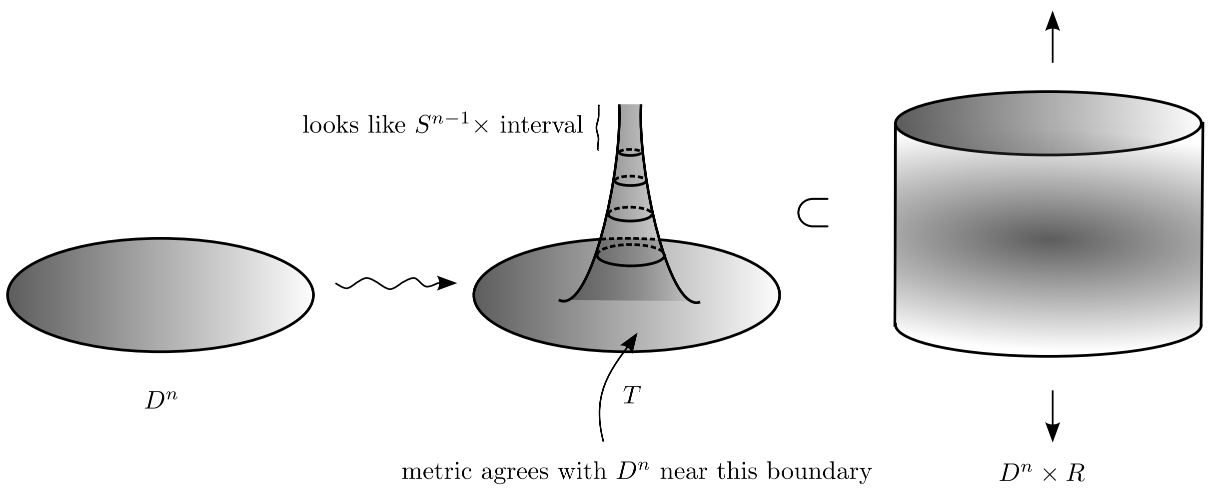


Figure 4: Constructing T

4. in the region $r_\infty < r < r_1$, γ is a graph $r = f(t)$ for a function f which is decreasing and (weakly) concave up

5. T has everywhere positive scalar curvature (this is the hard part!).

For now, assume we've constructed such a γ. We'll make T "standard" on the cylindrical end, using:

Lemma 3.2 *[GL] Let g_ϵ be the metric induced by $D^n \subset M^n$ on $S^{n-1}(\epsilon) \subset D^n$, and let $g_{0,\epsilon}$ be the standard round metric on the sphere of radius ϵ. Then*

$$(1/\epsilon^2)g_\epsilon \longrightarrow (1/\epsilon^2)g_{0,\epsilon} = g_{0,1}$$

in the $\mathcal{C}^2$ topology of metrics on S^{n-1}.

Then by picking r_∞ sufficiently small, we can ensure that the induced metric g_{r_∞} on $S^{n-1}(r_\infty)$ is isotopic (through p.s.c. metrics) to g_{0,r_∞} (recall that p.s.c. > 0 is an open condition). We can then use

Lemma 3.3 *[RS] Let ds_t^2, $0 \le t \le 1$ be a $\mathcal{C}^\infty$ family of p.s.c. metrics on a compact manifold X. Then for $a >> 0$, the metric $ds_{t/a}^2 + dt^2$ on $M \times [0, a]$ has p.s.c..*

Stated another way, "isotopic metrics are concordant".

Proof The scalar curvature on $M \times [0, a]$ with the metric $ds_{t/a}^2 + dt^2$ is given by

$$\kappa(x, t) = \kappa_{t/a}(x) + O(1/a^2),$$

where $\kappa_{t/a}$ is the scalar curvature on M with respect to the metric $ds_{t/a}^2$.

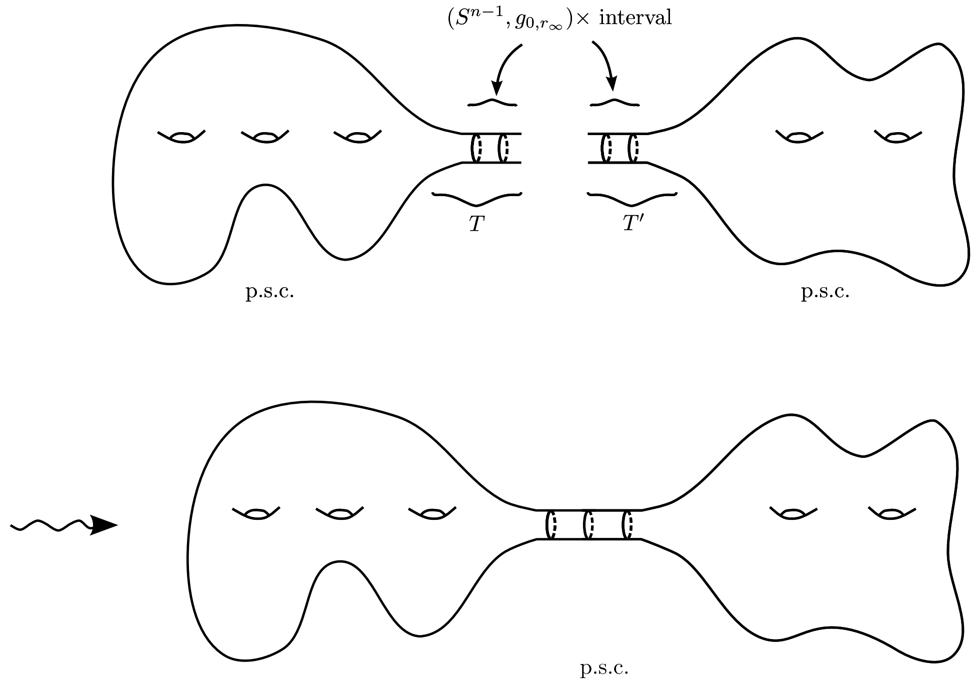


Figure 5: Performing a connected sum of p.s.c. manifolds

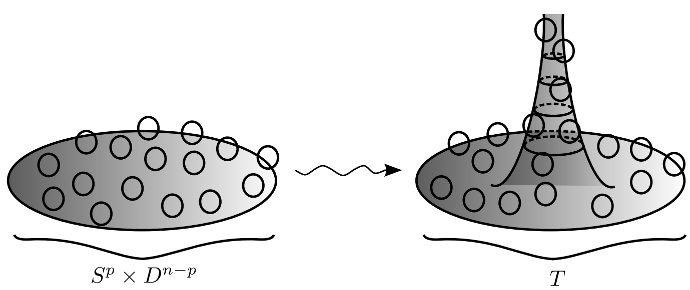


Figure 6: Performing a general surgery

Together these allow us to make T standard on the cylindrical end, so we can perform a connected sum (see Figure 5).

Example 3.4

Remark 3.5 *For higher dimensional surgeries, we can proceed in a similar manner: find an embedded $S^p \times D^{n-p} \subset M$ which is a geodesic tubular neighborhood of S^p, and deform it using the same γ. See Figure 6.*

T is constructed to have p.s.c. and to be a "standard" $(S^p, g_{0,1}) \times (S^{n-p-1}, g_{0,r_\infty}) \times$ interval on the cylindrical end. Namely, we set

$$T = \{(y, x, t) \in S^p \times D^{n-p}(\overline{r}) \ : \ (t, ||x||) \in \gamma\}.$$

On the cylindrical end, we then attach a "standard compatible" $D^{p+1}(1) \times S^{n-p-1}(r_\infty)$ of p.s.c..

Finally, we show how to construct γ. Recall the Gauss curvature equation: for a hypersurface $H^n \subset X^{n+1}$, with e, e' principal directions for the second fundamental form at a point $p \in H^n$, we have

$$R^{\text{sect}}_{H^n}(e, e'; p) = R^{\text{sect}}_{X^{n+1}}(e, e'; p) + \lambda_e \lambda_{e'}$$

(here R^{sect} denotes the sectional curvature and λ_e and $\lambda_{e'}$ are the corresponding principal curvatures.

Using this and some careful arguing, we can deduce the following formula for the scalar curvature of T:

$$\kappa_T = \kappa_M + O(1)\sin^2\theta + (q-1)(q-2)\frac{\sin^2\theta}{r^2} - (q-1)\frac{k\sin\theta}{r} - O(r)(q-1)k\sin\theta$$

where κ_T, κ_M denote scalar curvatures, k is the curvature of γ as a plane curve, and θ is the angle between γ and a vertical line.

Remark 3.6 *The condition $S^p = q \geq 3$ plays a role here, since our careful controlling of κ_T will rely on the fact that $q-1>0$ and $q-2>0$ in the equation above.*

Then for appropriate constants k_0, C, C' coming from the lower bound for κ_M (M is compact) and the $O(1)$ and $O(r)$ terms, we have

$$\kappa_T > k_0(q-1) - C\sin^2\theta(q-1) + (q-1)(q-2)\frac{\sin^2\theta}{r^2} - (q-1)\frac{k\sin\theta}{r} - C'r(q-1)k\sin\theta.$$

Then we have

$$\begin{aligned}\kappa_T > 0 \Longleftarrow \frac{k_0 r}{\sin\theta} - Cr\sin\theta + (q-2)\frac{\sin\theta}{r} - k - C'r^2k \geq 0 \\ \Longleftrightarrow (1+C'r^2)k \leq (q-2)\frac{\sin\theta}{r} + \frac{k_0 r}{\sin\theta} - Cr\sin\theta \end{aligned} \tag{1}$$

So we need to get from one end of the curve to the other, all the while satisfying (1).

- Firstly, pick a small $0 < \theta_0 < \arcsin(\sqrt{k_0/C})$, i.e. $\sin\theta_0 < \sqrt{k_0/C}$. Then for $\theta < \theta_0$, the last two terms of (1) are $\frac{k_0 r}{\sin\theta} - Cr\sin\theta > 0$. Then for a straight line of angle $\theta < \theta_0$, we have $k = 0$ and so (1) is satisfied. We begin by as in Figure 7.

- Next pick r_0 with $0 < r_0 < \min\left(\sqrt{\frac{1}{4C}}, \sqrt{\frac{1}{2C'}}\right)$, hence

$$\begin{aligned}(q-2)\frac{\sin\theta}{r} - Cr\sin\theta \geq \frac{3\sin\theta}{4r} \\ 1 + C'r^2 \leq 3/2.\end{aligned}$$

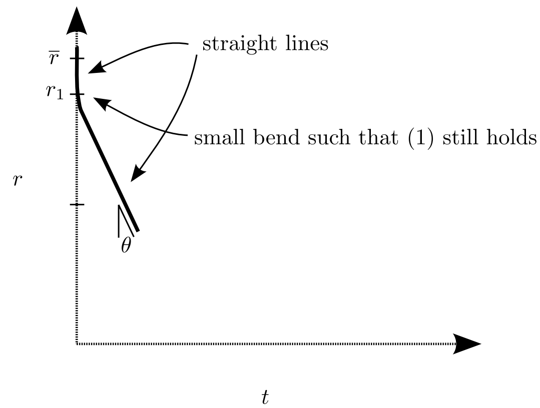


Figure 7: The first part of γ

Then for $r \le r_0$ we have

$$(1) \Longleftarrow k\frac{3}{2} \le \frac{3\sin\theta}{4r} + \frac{k_0 r}{\sin\theta}$$
$$\Longleftarrow k \le \frac{2}{3}\frac{3\sin\theta}{4r} = \frac{\sin\theta}{2r}.$$

So the curve $(t, f(t) = r)$ for $r \le r_0$ is only constrained by $k \le \frac{\sin\theta}{2r}$. Choosing "as much curvature as allowed", we get the following ODE:

$$k = \frac{\sin\theta}{2r} \rightsquigarrow f'' = \frac{1+(f')^2}{2f}$$

with $\sin\theta = \frac{1}{\sqrt{1+(f')^2}}$ and $k = \frac{f''}{(1+(f')^2)^{3/2}}$.

This can be solved explicitly (check!) as

$$f(t) = \frac{1}{C_1} + \frac{C_1}{4}(t - C_2)^2$$

for constants C_1, C_2.

So we follow our straight line of angle θ_0 until $r \le r_0$, then patch together with the above curve $r = f(t)$ for suitably chosen C_1, C_2 until r is very small, then patch with a straight horizontal line $r = r_\infty$. By construction (1) is always satisfied, so this completes the proof. ∎

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