The Classification of $(n-1)$-connected $2n$-manifolds

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1 Prologue

Our goal (following [Wal]):

Question 1.1 *For $2n \geq 6$, what is the diffeomorphic classification of $(n-1)$-connected $2n$-manifolds?*

This is the question Milnor was actually working as a long-term project during the 1950's when he "accidentally" discovered exotic spheres, as he explains in [Mil]. This was before the breakthrough of Smale and the Poincaré Conjecture still seemed completely out of reach. Somehow this was the "next easiest" problem in manifold topology and possibly ammenable to the extraordinary tools already developed by Thom and Hirzebruch.

First consider how we might construct examples in dimensions divisible by four.
dim $= 4$: Take a D^2 bundle over S^2 and attach a D^4 along the boundary to obtain a closed 4-manifold. Such bundles are classified by an element of $\pi_1 SO(2) \cong \mathbb{Z}$, i.e. the Euler number e. To attach D^4 we need the boundary to be S^3, which only occurs when $e = \pm 1$. Since $diff(S^3) \cong SO(4)$ (the Smale conjecture) is connected, there is an essentially unique way to attach D^4, and one can show that the result is always $\mathbb{CP}^2$.

dim $= 8$: Now we take a D^4 bundle over S^4 and try to attach D^8 to get a closed 8-manifold. Such a bundle is classified by its clutching function, an element of $\pi_3 SO(4) \cong \mathbb{Z} \oplus \mathbb{Z}$. Recall the special isomorphism

$$(SU(2) \times SU(2))\,/(\mathbb{Z}/2) \cong SO(4),$$

given by letting the first $SU(2) \cong S^3$ factor act on S^3 by left multiplication and the second factor act by right multiplication. A general element of $\pi_3 SO(4)$ can thus be written as $f_{ij} : S^3 \to SO(4)$, where

$$f_{i,j}(x)y = x^i y x^j.$$

Question 1.2 *When does $N^7_{i,j} := \partial(D^4 \times_{f_{i,j}} S^4)$ have the* homotopy type *of S^7?*

The answer turns out to be: precisely when $i + j = \pm 1$ (note: we will be cavalier with signs throughout).

Question 1.3 *When is $N^7_{i,1-i}$ diffeomorphic to S^7?*

If $\partial N^7_{i,1-i}$ were diffeomorphic to S^7, we could attach D^8 to form a closed 8-manifold $M^8_{i,j}$. By basic properties of Pontryagin classes, $p_1(M_{i,j}) = p_1(D^4 \times_{f_{i,j}} S^4) \in \mathbb{Z}$ (here we're implicitly evaluating on the generator in H_4). One can compute:

$$p_1(M_{i,j}) = p_1(M_{i,j}) = 2(i-j),$$

and hence

$$p_1(M_{i,1-i}) = 4(2i-1)^2.$$

On the other hand, the Hirzebruch signature theorem in this dimension gives

$$\mathrm{Sig}(M_{i,1-i}) = \frac{1}{45}(7p_2 - p_1^2).$$

Clearly $\mathrm{Sig}M = 1$ and therefore

$$p_2 = \frac{4(2i-1)^2 + 45}{7}.$$

- For $i = 1$, we get $p_2 = 7$, consistent with the fact that $M_{1,0} \cong \mathcal{HP}^2$, the quaternionic projective plane.
- For $i - 2$, we get $p_2(M_{2,-1}) = 81/7$, which is impossible. Conclusion: $M_{2,-1}$ doesn't exist, and $N^7_{2,-1}$ must be an exotic 7-sphere!

2 Reduction to handlebodies

We now fast-forward to the 1960's. Between Milnor's discovery and the present work of Wall, Smale proved the h-cobordism theorem. Actually, he really proved something stronger.

Theorem 2.1 *(Smale) Let M^b be a closed C^∞ manifold which is $(a-1)$-connected, with $b \geq 2a$ and $(b,a) \neq (4,2)$. Then M admits a self-indexing Morse function (i.e. critical values correspond to Morse indices) with*

- *a unique local minimum and a unique local maximum*
- *no index i critical points for $0 < i < a$ and $b - a < i < b$.*

Let M^{2n} be an $(n-1)$-connected manifold and let $M^{2n}_0 := M^{2n} \setminus \mathrm{int}D^{2n}$ denote the result after removing a top-dimensional disk. Applying the above result, we get

$$M^{2n}_0 \cong D^{2n} \cup H_1 \cup ... \cup H_r,$$

where the $H_i \cong D^n \times D^n$ are n-handles attached via disjoint embeddings $f_i : \partial D^n \times D^n \hookrightarrow \partial D^{2n}$. We refer to a manifold of this form as a "handlebody" Here the actual data of M^{2n}_0 consists of:

- disjointly embedded S^{n-1}'s with a "linking matrix" describing how they are pairwise linked
- a framing for each handle.

3 Intrinsic description of a handlebody

We now give a more intrinsic description of the handlebody M_0 and use this to classify handlebodies up to diffeomorphism. We have

- $H_n(M_0) \cong \mathbb{Z}^r$ is the only interesting homology group.
- H_n is equipped with an n-symmetric bilinear form[1], possibly degenerate.
- By Hurewicz: $H_n(M_0) \cong \pi_n(M_0)$.
- By a result of Haefliger: each element of $\pi_n(M_0)$ has an embedded representative, unique up to isotopy (at least for $n \geq 4$)
- Considering the clutching function of such a representative, we get a map

$$\alpha : H_n(M_0) \to \pi_{n-1}(SO(n)).$$

The above map α will play an important role. As we now explain, it satisfies some relations. First we need to recall a first classical items from algebraic topology.

- Associated to the fibration sequence $SO(n) \to SO(n+1) \to S^n$ there is a long exact sequence

$$\mathbf{1}_n \in \pi_n(S^n) \xrightarrow{\partial} \pi_{n-1}(SO(n)) \xrightarrow{S} \pi_{n-1}(SO(n+1))$$

- Let $J : \pi_r(SO(n)) \to \pi_{n+r}(S^n)$ be the J homomorphism. We quickly recall the definition. View an element $q \in \pi_r(SO(n))$ as a map $S^r \times S^{n-1} \to S^{n-1}$. Passing to the join, this induces a map $S^r * S^{n-1} \to S(S^{n-1})$, i.e. a map $S^{n+r} \to S^n$.
- Let $\mathcal{H} : \pi_{2n-1}(S^n) \to \mathbb{Z}$ be the Hopf invariant. Recall: the Hopf invariant of $f : S^{2n-1} \to S^n$ corresponds to the unique interesting cup product relation in $D^{2n} \cup_f S^n$.

Lemma 3.1 *The map α satisfies the relations*

- $x^2 = \mathcal{H}J\alpha(x) \qquad (\in \mathbb{Z})$
- $\alpha(x+y) = \alpha(x) + \alpha(y) + (x \cdot y)(\partial \mathbf{1}_n) \qquad (\in \pi_{n-1}(SO(n)))$

Note that α is *not* a homomorphism. One can view $\partial \mathbf{1}_n$ as some distinguished element of $\pi_{n-1}(SO(n))$ which satisfies $\mathcal{H}J\partial\mathbf{1}_n = 2$ for n even.

Proof sketch of lemma:

[1] Here n-symmetric means either symmetric or skew-symmetric, depending on the parity of n.

- $\mathcal{H}J : \pi_{n-1}(SO(n)) \to \mathbb{Z}$ can be identified with the map $\pi_{n-1}(SO(n)) \to \pi_{n-1}(S^{n-1})$ induced by evaluation at a point. Thus $\mathcal{H}J\alpha(x)$ is the obstruction to finding a section of the normal bundle of an embedded representative of $x \in H_n(M_0)$, which is precisely $x \cdot x \in \mathbb{Z}$.

- For two embedded spheres with corresponding normal bundles $\alpha(x)$ and $\alpha(y)$, we can form the ambient connected sum. It would have normal bundle representing $\alpha(x+y)$, except that it is generally only immersed. In fact, the signed count of self-interection points corresponds exactly to $x \cdot y$. We can now resolve each self-intersection point at the price of changing the normal bundle by $\partial \mathbf{1}_n$. Since the result is embedded, the second part of the lemma follows.

In summary, handlebodies are classified up to diffeomorphism by the data:

- H a free Abelian group
- $H \otimes H \to \mathbb{Z}$ an n-symmetric product
- $\alpha : H \to \pi_{n-1}(SO(n))$ a map satisfying

$$\begin{aligned} x^2 &= \mathcal{H}J\alpha(x) \quad (*) \\ \alpha(x+y) &= \alpha(x) + \alpha(y) + (xy)(\partial \mathbf{1}_n) \quad (**) \end{aligned}$$

We will refer to such data as an "algebraic handlebody".

Remark 3.2 *At first glance algebraic handlebodies might look like rather simple creatures, but note that, for n even, H is equipped with an integral quadratic form. Thus classifying algebraic handlebodies is at least as hard as classifying integral quadratic forms, which is an extremely difficult open problem.*

4 Closing the handlebody

Our goal is now to get a closed manifold M^{2n} by gluing a disk D^{2n} to the boundary of the handlebody M_0^{2n}. Note: for this to work we need ∂M_0^{2n} to be diffeomorphic to the standard smooth sphere S^{2n-1}.

Lemma 4.1

$$H_i(\partial M_0^{2n}) = \begin{cases} 0 & \text{for } i \neq 0, n-1, 2n-1, n \\ \operatorname{coker} \pi & \text{for } i = n-1 \\ \ker \pi & \text{for } i = n \end{cases}$$

Here $\pi : H \to H^*$ is given by $\pi(x)(y) = x \cdot y$ (i.e. the adjoint), where $H := H_n(M_0)$ and H^* is its dual vector space.

Corollary 4.2 ∂M_0 *is a homotopy sphere if and only if* π *is an isomorphism.*

Note that a simply connected homology sphere is a homotopy sphere by the Hurewicz theorem, and it is in fact homeomorphic to a sphere by the topological Poincaré conjecture.

Corollary 4.3 *Handlebodies with boundary homeomorphic a standard sphere are in one-to-one correspondence with algebraic handlebodies such that $H \otimes H \to \mathbb{Z}$ is* unimodular *(i.e. the matrix for it has unit determinant).*

Remark 4.4 *It is easy to check that performing a boundary connected sum of two handlebodies corresponds to taking a direct sum of two algebraic handlebodies.*

Corollary 4.5 *There is an additive map: $\{handlebodies\} \to \Gamma_{2n-1}$ which sends M_0 to ∂M_0.*

Here Γ_k denotes the group of exotic k-spheres under connected sums. There is a purely formal way of upgrading this additive map into a group homomorphism by turning the domain into a group. Let $\mathcal{G}_n$ denote the Grothendieck group of handlebodies, i.e. the free Abelian group generated by isomorphism classes of algebraic handlebodies, quotiented by all relations of the form $H_1 \oplus H_2 - H_1 - H_2$. We now have a group homomorphism

$$b : \mathcal{G}_n \to \Gamma_{2n-1}.$$

We'd like to understand the kernel of b.

Remark 4.6 *Even if ∂M_0 is diffeomorphic to S^{2n-1}, there is still ambiguity about* how *to glue in D^{2n} to form M. Following Wall, we will ignore this ambiguity in this talk, so M will only be determined up to connect-summing with an element of Γ_{2n}.*

To proceed, we must compute $\pi_{n-1}(SO(n))$. This is not quite a stable homotopy group but it was nonetheless computed by Kervaire. Unfortunately there are seven different cases to consider! For simplicity, we consider just two of them.

- *Case A:* $n \equiv 3, 5, 7 \bmod 8$, and $n \neq 3, 7$.
- *Case B:* $n \equiv 6 \bmod 8$.

4.1 Case A

In Case A, $\pi_{n-1}(SO(n)) \cong \mathbb{Z}/2$.

- $H \otimes H \to H$ is skew-symmetric and unimodular, hence classified by its rank r.
- $\alpha : H \to \mathbb{Z}/2$ satisfies $\alpha(x + y) = \alpha(x) + \alpha(y) + xy$.
- Let $e_1, e'_1, e_2, e'_2, \ldots$ be a symplectic basis for H (i.e. $e_i e'_i = -e'_i e_i = 1$ and $e_i e_j = 0$ otherwise).

It turns out that in this case α is completely classified by its $\mathbb{Z}/2$-valued *Arf invariant*, $\mathrm{Arf}(H, \alpha) := \sum_i \alpha(e_i)\alpha(e'_i) \bmod 2$. Checking that $\mathrm{Arf}(H, \alpha)$ is actually a well-defined invariant (independent of the chosen symplectic basis) is a straightforward but somewhat unenlightening algebraic exercise.

So in Case A, unimodular algebraic handlebodies are classified by $\mathrm{rank}(H)/2$ and $\mathrm{Arf}(H, \alpha)$. In particular, $\mathcal{G}_n \cong \mathbb{Z} \oplus \mathbb{Z}/2$.

4.2 Case B

In Case B, $\mathcal{H} \circ J : \pi_{n-1}(SO(n)) \to \mathbb{Z}$ is injective with index 2. The relation $(*)$ gives $x^2 = \mathcal{H}J\alpha(x)$. By the above properties of $\mathcal{H}J$, it follows that x^2 is always even. Moreover, α is uniquely determined by x^2.

Recall that an integral quadratic form $H \otimes H \to \mathbb{Z}$ such that $x \cdot x$ is always even is said to be "type II". We have that unimodular algeabraic handlebodies are in one-to-one correspondence with type II unimodular quadratic forms. Although the latter creatures are unclassified, one can still establish the following theorem using some deep facts about quadratic forms.

Theorem 4.7 *$\mathcal{G}_n \cong \mathbb{Z} \oplus \mathbb{Z}$, given by $\left(\frac{rank - sig}{2}, \frac{sig}{8}\right)$. Here sig denotes the signature, which is always divisible by 8 for type II quadratic forms.*

In particular, the quantities $\frac{\text{rank} - \text{sig}}{2}$ and $\frac{\text{sig}}{8}$ determine whether or not M_0 can closed up.

5 Obstruction to closing

In Case A, Kervaire showed that ∂M_0 is a standard sphere if and only if $\text{Arf} = 0$.

In Case B, the boundary is standard if and only the following quantity vanishes:

$$\text{sig}/8 \mod \frac{2^{n-3}(2^{n-1}-1)B_{n/2}j_{n/2}a_{n/2}}{n},$$

where

- B_i = Bernoulli number
- $a_i = 2$ for i odd and 1 for i even
- j_i is the order of the image of the *stable* J-homomorphism $J : \pi_{4i-1}(SO) \to \pi^S_{4i-1}$.

Remark 5.1 *The quantity j_i was only computed about a decade after Wall's paper was published.*

Remark 5.2 *A unimodular algebraic handlebody always uniquely closes up to an $(n-1)$-connected PL manifold. On the other hand, when the above obstruction quantity is nonzero, we get a PL manifold with no smooth structure (in fact, it is not even homotopy equivalent to a smooth manifold!). The first such example was discovered with $n = 10$ by Kervaire around 1960. Namely, he found a PL manifold with nonzero Arf invariant, and hence no smooth structure.*

References

[Mil] John Milnor. Classification of $(n-1)$-connected $2n$-dimensional manifolds and the discovery of exotic spheres.

[Wal] CTC Wall. Classification of $(n-1)$-connected $2n$-manifolds. *Annals of Mathematics* (1962), 163–189.