# Mirror Symmetry: Introduction to the B Model

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## 1 Introduction

Recall that mirror symmetry predicts the existence of pairs $X, \check{X}$ of Calabi-Yau manifolds whose Hodge diamonds are mirror images of each other, i.e. $H^q(X, \Lambda^p TX) \simeq H^q(\check{X}, \Omega^p \check{X})$. In fact, mirror symmetry reflects much more than just the Hodge structures, and we also get an isomorphism between the "Yukawa couplings" on $H^1(X, TX)$ and $H^{1,1}(\check{X})$, which give product structures $H^1(X, TX) \otimes H^1(X, TX) \to H^1(X, TX)$ and $H^{1,1}(\check{X}) \otimes H^{1,1}(\check{X}) \to H^{1,1}(\check{X})$. The coupling on $H^1(X, TX)$ is defined in terms of Gromov-Witten invariants of $X$ and often contains deep enumerative information about $X$. For example, when $X$ is the quintic threefold, it is much easier to calculate the Yukawa couplings for the mirror $\check{X}$, and mirror symmetry then gives astonishing formulas for the number of degree $d$ rational curves on $X$, for *all* $d$.

We aim to give a more refined statement of mirror symmetry in terms of the complex and Kahler moduli spaces of $X$ and $\check{X}$ respectively. Let $\mathcal{M}_{\text{cx}}(X)$ and $\mathcal{M}_{\text{kah}}(\check{X})$ denote the complex and Kahler moduli spaces of $X$ and $\check{X}$ respectively. The former is defined to be the moduli space of complex structures on $X$. Assuming $h^{2,0}(\check{X}) = 0$, the latter can be defined in terms of the Kahler cone $K(\check{X}) \subset H^2(\check{X}, \mathbb{R})$ consisting of all Kahler classes. Namely, we define $\mathcal{M}_{\text{kah}}(\check{X})$ to be $K_{\mathbb{C}}(V)/\text{Aut}(\check{X})$, where $K_{\mathbb{C}}(\check{X})$ is the "complexified Kahler space" given by

$$K_{\mathbb{C}}(\check{X}) = \{\omega \in H^2(\check{X}, \mathbb{C}) \mid \text{Im}(\omega) \in K(\check{X})\}/\text{im}H^2(\check{X}, \mathbb{Z}).$$

The goal of this talk is to discuss the following statement of mirror symmetry:

**Conjecture 1.1** *Let $\mathcal{X} \to (D^*)^s$ be a family of Calabi-Yau 3-folds with a large complex structure limit (LCSL) point at 0. Then there is another Calabi -Yau 3-fold $\check{X}$ and a choice of bases*

$$\begin{aligned} \alpha_0, ..., \alpha_s, \beta_0, ..., \beta_s \quad &\textit{for} \quad H_3(X, \mathbb{Z}) \\ e_1, ..., e_3 \quad &\textit{on} \quad H^2(\check{X}, \mathbb{Z}) \end{aligned}$$

*giving rise to a locally defined map*

$$\begin{aligned} m : \mathcal{M}_{cx}(X) &\to \mathcal{M}_{kah}(\check{X}), \\ (q_1, ..., q_s) &\mapsto (\check{q}_1, ..., \check{q}_s), \end{aligned}$$

*such that the Yukawa couplings match:*

$$\left\langle \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_k} \right\rangle_p = \left\langle \frac{\partial}{\partial \check{q}_i}, \frac{\partial}{\partial \check{q}_j}, \frac{\partial}{\partial \check{q}_k} \right\rangle_{m(p)}.$$

Here the LCSL condition essentially corresponds to a complexified Kahler form $[B + i\omega]$ for $\omega$ sufficiently positive. We will show that the bases give rise to local coordinates $q_i$ and $\check{q}_i$ on $\mathcal{M}_{\text{cx}}(X)$ and $\mathcal{M}_{\text{kah}}(\check{X})$ and thus the map $m$.

**Example 1.2** *Consider the family of elliptic curves*

$$C_t = \{y^2 z = x^3 + x^2 z - tz^3\} \subset \mathbb{CP}^2.$$

*Note that $C_t$ is smooth for $t \neq 0$, and $C_0$ has a nodal singularity. As $t$ travels around the origin, the monodromy is a Dehn twist around a vanishing cycle. The induced map on homology $H_1(C_{t_0}) \simeq \mathbb{Z}^2 = \langle a, b \rangle$ is given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.*

*Using period integrals, we replace the ad hoc parameter $t$ with a natural local coordinate $q$ for the family. First, equip each $C_t$ with a holomorphic volume form $\Omega_t$ such that, for each $t$,*

$$\int_a \Omega_t = 1.$$

*Now let*

$$\tau(t) = \int_b \Omega_t.$$

*Abstractly, we have $C_t \simeq \mathbb{C}/(\mathbb{Z} + \tau(t)\mathbb{Z})$. As $t$ goes around the origin, $\tau(t)$ goes to $\tau(t) + \int_a \Omega_{t_0} = \tau(t) + 1$. Therefore $q(t) := e^{2\pi i \tau(t)}$ is single-valued and gives a local coordinate for the family. As $t \to 0$, Im $\tau(t) \to \infty$ and $q(t) \to 0$. This is an example of a LCSL.*

# 2 Deformations of Complex Structures

For a given Calabi-Yau manifold, we would like to study the local structure of the moduli space of complex structures. To start, let $(X, J)$ be an almost complex manifold. Recall that we have a decomposition

$$TX \otimes \mathbb{C} = TX_J^{1,0} \oplus TX_J^{0,1}$$

of the complexified tangent bundle of $X$ into the $i$ and $-i$ eigenspaces respectively of $J$. Note that for $v \in TX \otimes \mathbb{C}$, we have $v = v_J^{1,0} + v_J^{0,1}$, with $v_J^{1,0} = \frac{1}{2}(v - iJv) \in TX^{1,0}$ and $v_J^{0,1} = \frac{1}{2}(v + iJv) \in TX_J^{1,0}$. Similarly, we have decompositions

$$T^*X \otimes \mathbb{C} = T^*X_J^{1,0} \oplus T^*X_J^{0,1},$$
$$\Lambda^k T^*X = \oplus_{p+q=k}\Omega_J^{p,q}(X).$$

Now for $J'$ another almost complex structure close to $J$, we can view

$$\Omega_{J'}^{1,0} \subset T^*X \otimes \mathbb{C} = \Omega_J^{1,0} \oplus \Omega_J^{0,1}$$

as the graph of a linear map $s : \Omega_J^{1,0} \to \Omega_J^{0,1}$. Conversely, for such an $s$ sufficiently small we can set

$$\Omega_{J'}^{1,0} := \mathrm{graph}(s),$$
$$\Omega_{J'}^{0,1} := \overline{\Omega_{J'}^{1,0}},$$

and then define the action of $J'$ to be multiplication by $i$ on $\Omega_{J'}^{1,0}$ and by $-i$ on $\Omega_{J'}^{0,1}$. Note that $s$ can also be viewed as a section of

$$(\Omega_J^{1,0})^* \otimes \Omega_J^{0,1} \simeq T_J^{1,0} \otimes \Omega_J^{0,1}.$$

Of course, we want the deformation $J'$ to be *integrable*. Recall that the almost complex structure $J$ is integrable if and only if we have

$$[TX^{1,0}, TX^{1,0}] \subset TX^{1,0}.$$

Note that the Dolbeaut complex for $TX_J^{1,0}$ on $(X, J)$, namely $\oplus_q \Omega_X^{0,q} \otimes TX^{1,0}$, carries a Lie bracket given by

$$[\alpha \otimes v, \alpha' \otimes v'] := (\alpha \wedge \alpha') \otimes [v, v'].$$

Using local coordinates, one can easily show:

**Proposition 2.1** *$J'$ is integrable if and only if $\overline{\partial}s + \frac{1}{2}[s, s] = 0$.*

Recall that we also need to quotient by Diff$(X)$. Let $\phi$ be a diffeomorphism of $X$ which is close to the identity. We first remark that for $\{z_i\}$ local holomorphic coordinates for $(X, J)$, one can check that a basis for the $(1, 0)$-forms corresponding to $J'$ is given by

$$\{dz_i - s(dz_i)\}.$$

Decomposing $d\phi$ into parts which commute and anticommute with $J$:

$$d\phi = \partial\phi + \overline{\partial}\phi,$$

we therefore have

$$\phi^* dz_i = dz_i \circ \partial\phi + dz_i \circ \overline{\partial}\phi = (dz_i \circ \partial\phi) \circ (\mathrm{Id} + (\partial\phi)^{-1}\overline{\partial}\phi),$$

and hence corresponding to $\phi^* J$ we have

$$s = -(\partial\phi)^{-1}\overline{\partial}\phi.$$

Putting this all together, consider a deformation $J(t)$ of $J$ with $J(0) = J$, corresponding to $s(t) = ts_1 + t^2 s_2 + t^3 s_3 + ... \in \Omega^{0,1}(X, TX^{1,0})$. Assuming the family $J(t)$ is integrable, the equation

$$\overline{\partial}s(t) + \frac{1}{2}[s(t), s(t)] = 0$$

gives

$$\frac{1}{2}[ts_1 + t^2 s_2 + ..., ts_1 + t^2 s_2 + ...] + t\overline{\partial}s_1 + t^2\overline{\partial}s_2 + ... = 0.$$

In particular, the first order part in $t$ gives

$$\overline{\partial}s_1 = 0.$$

On the other hand, if $\phi_t$ is a family of diffeomorphisms of $X$ with $\phi_0 = \mathrm{Id}$, we have

$$\left.\frac{d}{dt}\right|_{t=0} (-(\partial\phi_t)^{-1}\overline{\partial}\phi_t) = -\left.\frac{d}{dt}\right|_{t=0} (\overline{\partial}\phi_t) = -\overline{\partial}v,$$

where $v \in \Gamma(TX)$ is the vector field generating the family $\phi_t$. In other words, first order deformations of $(X, J)$ correspond to

$$\frac{\mathrm{Ker}(\overline{\partial} : \Omega^{0,1}(X, TX^{1,0}) \to \Omega^{0,2})}{\mathrm{Im}(\overline{\partial} : C^\infty(X, TX^{1,0}) \to \Omega^{0,1})} = H^1(X, TX^{1,0}).$$

Moreover, if $(X^n, J)$ is Calabi-Yau with $\Omega$ a holomorphic volume form, then $\Omega$ gives an identification $TX^{1,0} \simeq \wedge^{n-1}T^*X$, and therefore we have

$$H^1(X, TX^{1,0}) \cong H^1(X, \wedge^{n-1}T^*X) \cong H^{n-1,1}(X, J).$$

A priori there may be obstructions to finding the higher order parts of $s(t)$. Namely, we must have

$$\begin{aligned} \overline{\partial}s_2 + \frac{1}{2}[s_1, s_2] &= 0, \\ \overline{\partial}s_3 + [s_1, s_2] &= 0, \\ \overline{\partial}s_4 + [s_1, s_3] + \frac{1}{2}[s_2, s_2]&, \end{aligned}$$

etc, and so there are obstructions lying in $H^2(X, TX)$. Happily, we have

**Theorem 2.2** *(Bogomolov-Tian-Todorov) For $X$ a compact Calabi-Yau with $H^0(X, TX) = 0$, deformations of $X$ are unobstructed.*

## 3 The Hodge Bundle

Partly for convenience, we now focus on three dimensional Calabi-Yau manifolds. In this case, the cohomology groups $H^3(X;\mathbb{C})$ glue together to form a bundle $\mathcal{H}$, the *Hodge bundle*, over the moduli space of complex structures. The Calabi-Yau forms, unique up a constant, form a line sub-bundle of the Hodge bundle. Moreover, by declaring integer cohomology classes to be flat sections, we get a connection the Hodge bundle, the *Gauss-Manin connection.*

It turns out that the position of the Calabi-Yau form (up to scaling) in $H^3$ (locally) determines the complex structure. We can think of the Calabi-Yau form as determining a point in $\mathcal{P}^{h^3-1}$, where $h^3$ is the dimension of $H^3$. Since the dimension of $\mathcal{M}_{\text{cx}}(X)$ is only $h^{2,1} = \frac{1}{2}h^3 - 1$, this description of the complex structure is redundant. To find sharper coordinates on $\mathcal{M}_{\text{cx}}(X)$, we first define a natural Hermitian metric $(\cdot,\cdot)$ on $\mathcal{H}$ by

$$(\theta,\eta) = i\int \theta \wedge \overline{\eta}, \qquad \theta,\eta \in H^3(M,\mathbb{C}).$$

Then we can find a "symplectic basis" of real integer three-forms $\alpha_a, \beta^b,\ a,b = 1,...,h^3/2$, such that

$$\begin{aligned}(\alpha_a,\alpha_b) = (\beta^a,\beta^b) = 0 \\ (\alpha_a,\beta^b) = i\delta_a^b,\end{aligned}$$

with dual basis $A^a, B_b,\ a,b = 1,...,h^3/2$.

## 4 Periods and Coordinates on Moduli Space

We first discuss the coordinates on $\mathcal{M}_{\text{kah}}(\check{X})$. If $e_i$ is a basis for $H^2(\check{X},\mathbb{Z})$ with each $e_i$ in the Kahler cone, we get local coordinates $\check{q}_i$ on $\mathcal{M}_{\text{kah}}(\check{X})$ by setting

$$\begin{aligned}[B + i\omega] = \sum_i \check{t}_i e_i, \\ \check{q}_i = \exp(2\pi i\check{t}_i) \in \mathbb{C}^*.\end{aligned}$$

Now to construct coordinates on $\mathcal{M}_{\text{cp}}(X)$, let $\Omega$ be a Calabi-Yau form. We consider the "period integrals"

$$z^a = \int_{A^a}\Omega, \qquad \omega_b = \int_{B^b}\Omega.$$

It turns out that the complex structure is locally determined by just the $z^a$. In fact, there are $h^3/2$ of them, while the moduli space is only of dimension $h^{2,1} = h^3/2 - 1$. However, the Calabi-Yau form $\Omega$ is only well-defined up to scaling, so we view the $z^a$ as homogenous local coordinates.

Now suppose $\mathcal{X} \to (D^*)^s$ is a *LCSL*. If $h^{n-1,1} = s = 1$, this equivalent to the monodromy $\phi_* \in \mathrm{Aut}(H^n(X_{t_0}, \mathbb{Z}))$ around 0 being *maximally unipotent*, i.e.

$$(\phi_* - \mathrm{Id})^k = 0$$

for $k = n+1$ but not for $k < n+1$. For $s > 1$ there is a more involved definition in terms of the Jordan decompositions of the various induced monodromy actions on $H^n(X_{t_0}, \mathbb{Z}))$ corresponding to loops in $(D^*)^s$. Let $\Omega$ be such that $\int_{B_0} \Omega = 1$. Then setting $q_i = \exp(2\pi i \omega_i)$ defined *canonical coordinates* $q_i$ on $(D^*)^s$ (which of course are only canonical after a basis is chosen).

# 5 Yukawa Couplings

On the $A$ side, the Yukawa couplings are given as follows. For $\omega_1, \omega_2, \omega_3 \in H^{1,1}(\check{X})$,

$$\langle \omega_1, \omega_2, \omega_3 \rangle := \int_X \omega_1 \wedge \omega_2 \wedge \omega_3 + \sum_{0 \neq \beta \in H_2(X,\mathbb{Z})} n_\beta \int_\beta \omega_1 \int_\beta \omega_2 \int_\beta \omega_3 \frac{e^{2\pi i \int_\beta \omega}}{1 - e^{2\pi i \int_\beta \omega}},$$

where $n_\beta$ is defined in terms of Gromov-Witten invariants and is roughly the "number of holomorphic spheres in $\check{X}$ of class $\beta$". The numbers $n_\beta$ contain deep information about the arithmentic properties of $\check{X}$.

On the $B$ side, the Yukawa coupling for $\langle \theta_1, \theta_2, \theta_3 \rangle \in H^1(X, TX)$ is given by

$$\langle \theta_1, \theta_2, \theta_3 \rangle := \int_X \Omega \wedge (\theta_1 \cdot \theta_2 \cdot \theta_3 \cdot \Omega),$$

using the composition

$$S^3 H^1(X, TX) \otimes H^0(X, \Omega^3 X) \to H^3(X, \Lambda^3 TX \otimes \Omega^3 X) \simeq H^3(X, \mathcal{O}_X) \simeq H^{0,3}(X).$$

Equivalently, this is

$$\int_X \Omega \wedge (\nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega),$$

where $\nabla$ is the Gauss-Manin connection.

**Remark 5.1** *Using $z^a$ and $\omega_b$, we define the "prepotential" $\mathcal{G}$:*

$$\mathcal{G} := z^a \omega_a.$$

*Then using $\mathcal{G}$ we can recover the Yukawa couplings*

$$\kappa_{a,b,c} = \langle \chi_a, \chi_b, \chi_c \rangle,$$

*where $\chi_a$ is the $(2,1)$ part of $\partial_a \Omega$ (considered as an element of $H^1(TM)$, by*

$$\kappa_{a,b,c} = \partial_a \partial_b \partial_c \mathcal{G}.$$

# 6 Mirror Symmetry for the Quintic Threefold

Let $V \subset \mathbb{CP}^4$ be a smooth quintic hypersurface, i.e. the zero set of a homogeneous degree 5 polynomial. Then $V$ is Calabi-Yau. If $H$ denotes the hyperplane class, the $A$-model Yukawa coupling formula becomes

$$\langle H, H, H\rangle = 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d}$$

where $q = \exp(2\pi i \int_l \omega)$, $l$ is a line in $V$, and $\omega = B + iJ$ is a complexified Kahler class on $V$. The first few values of $n_d$ are given by

$$\begin{aligned} n_1 &= 2,875 \\ n_2 &= 609,250 \\ n_3 &= 317,206,375 \\ n_4 &= 242,467,530,000, \end{aligned}$$

etc.

**Remark 6.1** *Although $n_d$ is indeed the number of rational curves of degree $d$ in $V$ for $d \leq 9$, in general the enumerative content of $n_d$ is more subtle. In particular, $n_{10}$ does* not *give the number of degree 10 rational curves on $V$, as double covers of nodal rational curves contribute more than expected.*

One of the most striking early applications of mirror symmetry to mathematics was the computation of the above expression using the $B$-model couplings on the mirror of $V$. In this last section we explain the mirror of $V$ and how to get the mirror map.

Recall that the Hodge diamond of $V$ is given by $h^{1,1}(V) = 1$ and $h^{2,1}(V) = 101$. Therefore the mirror $\check{V}$ should satisfy $h^{2,1}(\check{V}) = 1$ and hence live in a one-parameter family of Calabi-Yau manifolds. We describe it as a resolution of singularities of a family of hypersurfaces in $\mathbb{CP}^4/G$, where

$$G = \{(a_1, ..., a_5) \in (\mathbb{Z}/5)^5 \mid \sum_i a_i \cong 0 \bmod 5\}/(\mathbb{Z}/5),$$

and $g = (a_1, ..., a_5)$ acts by

$$g \cdot (x_1, ..., x_5) = (\mu^{a_1} x_1, ..., \mu^{a_5} x_5)$$

for $\mu = \exp(2\pi i/5)$.

With parameter $\psi$, the hypersurfaces are defined by

$$x_1^5 + ... + x_5^5 + \psi x_1 x_2 x_3 x_4 x_5 = 0.$$

These hypersurfaces inherit singularties from $\mathbb{CP}^4/G$, and we let $\check{V}_\psi$ be the result after simultaneously resolving the singularities. Now we observe that the map

$$(x_1, ..., x_5) \mapsto (\mu^{-1}x_1, x_2, ..., x_5)$$

induces an isomorphism $\check{V}_\psi \simeq \check{V}_{\mu\psi}$, and $\psi^5$ is well-defined on the moduli space of complex structures on $\check{V}_\psi$. We set $x = \psi^{-5}$ as a local coordinate for the complex moduli. The singularities of $\check{V}_\psi$ occur for $\psi = -5\mu^i$, $0 \leq i \leq 4$, and for $\psi = \infty$. In terms of $x$, they occur for $x = -5^{-5}$ and $x = 0$.

Now since $H$ generates $H^2(V, \mathbb{Z})$, we can write any complexified Kahler class as $\omega = tH$ for $t$ in the upper half plane. Since $K_{\mathbb{C}}(V)$ is the quotient of $K_{\mathbb{C}}(V)$ by $H^2(V, \mathbb{Z})$, setting $q = \exp(2\pi i t)$ gives an isomorphism

$$q : K_{\mathbb{C}}(V) \simeq \Delta^*.$$

It turns out that the limit point $0 \in \Delta$ corresponds to a LCSL. In the complex structure moduli space, the point $x = 0$ has maximally unipotent monodromy and therefore should be the image of $q = 0$.

However, the mirror map $\mathcal{M}_{\text{kah}}(V) \to \mathcal{M}_{\text{cx}}(\check{V})$ is *not* given by $q = x$. Rather, we put a coordinate $\check{q}$ on $\check{V}$ as follows. We claim that there is a minimal integral vanishing cycle $\gamma_0$ near $x = 0$ such that $\gamma_0$ is invariant under monondromy, and that there is a minimal integral cycle $\gamma_1$ which transforms under the monodromy about $x = 0$ by $\gamma_1 \mapsto \gamma_1 + \gamma_0$. Then for $\Omega$ a holomorphic 3-form, monodromy around $x = 0$ gives the transformation

$$\int_{\gamma_1} \Omega / \int_{\gamma_0} \Omega \mapsto \int_{\gamma_1} \Omega / \int_{\gamma_0} \Omega + 1,$$

and we set

$$\check{q} = \exp(2\pi i \int_{\gamma_1} \Omega / \int_{\gamma_0} \Omega).$$

Then $q$ and $\check{q}$ correspond under the mirror map.

**Remark 6.2** *In order to equate the Yukawa couplings, the next step would be to find an expression for $\check{q}$ in terms of $x$. To do this, we can use the fact that $\int_{\gamma_0} \Omega$ and $\int_{\gamma_1} \Omega$ are periods and therefore satisfy a Picard-Fuchs equation of the form*

$$y'''' + f_1 y''' + f_2 y'' + f_3 y' + f_4 y = 0,$$

*for the $f_i$ functions of $x$ and differentation taken with respect to $x$. This comes from the fact that $h^3(\check{V}) = 4$ and hence any 5 sections of the Hodge bundle must be linearly dependent. Using techniques from ordinary differential equations, we ultimately find that*

$$\check{q} = -(x - 770x^2 + ...).$$

# References

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