Introduction to the h-Principle

Kyler Siegel

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1 Introduction

The goal of this talk is to give a basic introduction to h-principle and some of the ideas involved in its proof, with a primary focus on Gromov's h-principle for contact structures on open manifolds. To illustrate the h principle, we begin with an example, one of the first h-principles discovered.

Theorem 1.1 *Smale, Hirsch '59 The natural map*

$$\{immersions\ V \to W\} \hookrightarrow \{fiberwise\ injective\ bundle\ maps\ TV \to TW\},$$

is a homotopy equivalence, where V^n and W^q are smooth manifolds with $n < q$.

Note that, in particular, this theorem characterizes immerions up to regular homotopy in terms of homotopy classes of bundle maps.

Remark 1.2 *The condition $n < q$ is essential and is a typical hypothesis in h-principles.*

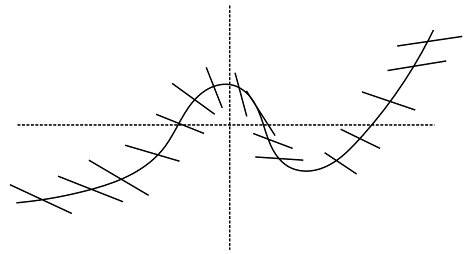


Figure 1: A typical section of $J^1(\mathbb{R},\mathbb{R})$

2 Jet bundles and holonomic sections

Given a fiber bundle $X \to V$, we can construct another fiber bundle $J^r(X) \to X$ (which is evidently also a bundle over V), with the fiber of $J^r(X)$ over $p \in V$ given by

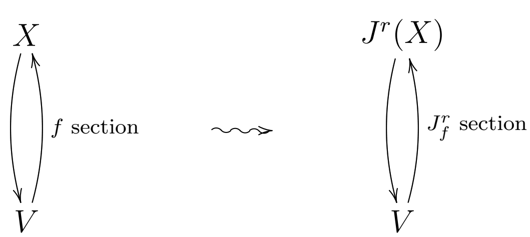
$$\{\text{sections } \in C^\infty(\mathcal{O}p(p), X)\}/\sim,$$

where for $F : U \to X$, $G : V \to X$, we declare $F \sim G$ if, in local coordinates, F and G have the same rth order Taylor polynomials at p. Recall that $\mathcal{O}p(p)$ denotes a small, unspecified neighborhood of p.

Notation 2.1 *For smooth manifolds V, W, we set $J^r(V,W) := J^r(V \times W)$, thinking of $V \times W$ as the trivial bundle $V \times W \to V$.*

Example 2.2
- $J^0(X) = X$
- $J^1(V,\mathbb{R}) = \mathbb{R} \times T^*V$
- $J^1(V,W) = \{(p,q,P) \ : \ p \in V, q \in W, P \subset T_pV \times T_qW, P \cap T_qW = \{0\}\}$
- *A typical section of $J^1(\mathbb{R},\mathbb{R})$ is portrayed in Figure 1.*
- $J^r(\mathbb{R}^n,\mathbb{R}^q) \approx \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^{qd_1} \times \mathbb{R}^{qd_2} \times \ldots \times \mathbb{R}^{qd_r}$, *where* $d_i = \dfrac{(n+i-1)!}{(n-1)!i!}$.

There is a tautological association



given essentially by taking derivatives.

Definition 2.3 *A section of* $J^r(X) \to V$ *is called* holonomic *if* $F = J^r_{bs\,F}$*, where* $bs\ F$ *denotes the underlying section of* $X \to V$*.*

Notation 2.4 *We will use* bs *to denote the projection map* $bs : J^r(X) \to X$ *as well as the induced map on sections* $bs : \sec(J^r(X)) \to \sec(X)$*.*

Example 2.5
- *An arbitrary section of* $J^3(\mathbb{R}, \mathbb{R})$ *is of the form* $x \mapsto (x, f(x), g(x), h(x), k(x))$*.*
- *A holonomic section of* $J^3(\mathbb{R}, R)$ *is of the form* $x \mapsto (x, f(x), f'(x), f''(x), f'''(x))$*.*

3 Holonomic approximation

Our desire is to approximate a section F of $J^r(X) \to V$ by a *holonomic* one.

Futile Example 3.1 *Let* F *be the section of* $J^1(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ *given by* $F(x) = (x, x, 0)$*. A holonomic approximation would be a function* $f(x)$ *with* $|f'(x)| < \epsilon$ *and* $|f(x) - x| < \epsilon$*.*

Example 3.2 Near a point $p \in V$*, we can always approximate* $F|_{\mathcal{O}p(p)}$ *by the Taylor polynomial specified by* $F(p)$*, which is holonomic.*

Question 3.3 *Can we always holonomically approximate* F *near a* 1*-cell?*

Futile Example 3.4 *Consider the section* F *of* $J^1(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}^2$ *given by*

$$F(x_1, x_2) = (x_1, x_2, f(x_1, x_2), 0, 0),$$

with $f(x_1, x_2) = x_1$*. Consider the problem of finding a holonomic approximation of* F *near* $I := \{(x_1, 0) \ : \ 0 \le x_1 \le 1\} \subset \mathbb{R}^2$*. A little thought shows that this is impossible, for essentially the same reason as before.*

Key idea: if we are allowed to replace the 1-cell I by some C^0-close 1-cell I', then we *can* holonomically approximate F on $\mathcal{O}p(I')$!

Theorem 3.5 *(Holonomic Approximation) Let* $A \subset V$ *be a polyhedron of positive codimension and let* $F : \mathcal{O}p(A) \to J^r(X)$ *be a section. Then for any* $\epsilon > 0$*, there exists an* ϵ C^0*-small diffeotopy* $h^t : V \to V$*,* $t \in [0, 1]$*, and a holonomic section* $\tilde{F} : \mathcal{O}p\ h^1(A) \to J^r(X)$ *such that dist*$(\tilde{F}(v), F(v)) < \epsilon$ *for all* $v \in \mathcal{O}p\ h^1(A)$*.*

There is also *parametric* version, which is essentially the same result but with some extra parameters floating around.

Theorem 3.6 *(Parametric Holonomic Approximation) Let* $A \subset V$ *be a polyhedron of positive codimension and suppose we have sections*

$$F_z : \mathcal{O}p\ A \to J^r(X), \qquad z \in I^m$$

with F_z *holonomic for all* $z \in \mathcal{O}p\ \partial I^m$ *(here* I^m *denotes the m-dimensional unit cube). Then for any* $\epsilon > 0$*, there exists a family of* ϵ C^0*-small diffeotopies* $h^t_z : V \to V$*,* $t \in [0, 1]$*,* $z \in I^m$*, and holonomic sections* $\tilde{F}_z : \mathcal{O}p\ h^1_z(A) \to J^r(X)$*,* $z \in I^m$*, such that*

- $h_z^t = id_V$ *and* $\tilde{F}_z = F_z$ *for all* $z \in \mathcal{O}p\, \partial I^m$
- *dist*$(\tilde{F}_z(v), F_z(v)) < \epsilon$ *for all* $v \in \mathcal{O}p\, h_z^1(A)$*,* $z \in I^m$*.*

Remark 3.7 *There is also a* relative *version, where* F *is already holonomic near a subpolyhedron* B*, and* h *is the identity and* $\tilde{F} = F$ *on* $\mathcal{O}p(B)$*.*

4 Applications

Let V^{2n+1} be an open manifold (i.e. every component is either noncompact or has nonempty boundary). Here is the main result we are interested in.

Theorem 4.1 *(Gromov) The natural map*

$$\{\text{cooriented contact structures on } V\} \hookrightarrow \{\text{cooriented almost contact structures on } V\}$$

is a homotopy equivalence.

Here the set on the right hand side can be identified with the set of pairs (ξ_+, ω) such that ξ_+ is a cooriented hyperplane distribution on V and ω is a positive conformal class of symplectic structures on ξ_+.

Proof There is a bundle homomorphism $D : J^1(\Lambda^1 V) \to \Lambda^2 V$, called the "symbol of d", such that the composition

$$\sec(\Lambda^1 V) \longrightarrow \sec(J^1(\Lambda^1 V)) \longrightarrow \sec(\Lambda^2 V)$$

of J^1 and the map $\tilde{D}$ on sections induced by D gives the usual exterior derivative

$$d : \sec(\Lambda^1 V) \to \sec(\Lambda^2 V).$$

Remark 4.2 *In local coordinates, the fiber of* $J^1(\Lambda^1 V)$ *is* $M_n(\mathbb{R})$*, the fiber of* $\Lambda^2 V$ *is the set of skew-symmetric* $n \times n$ *matrices, and* $D(A) = A - A^T$*.*

Remark 4.3 *Given* $(\alpha, \beta) \in \Lambda^1 V \oplus \Lambda^2 V$*, we can always find* $F \in \sec(J^1(\Lambda^1 V))$ *such that* $(\alpha, \beta) = (bs\ F, \tilde{D}F)$*. In particular, a cooriented almost contact structure gives rise to a section of* $J^1(\Lambda^1 V)$ *(and in fact the space of such lifts is contractible).*

Consider the projection $bs : J^1(\Lambda^1 V) \to \Lambda^1 V$. Let $\mathcal{R}_{\text{cont}} \subset J^1(\Lambda^1 V)$ be given by

$$\mathcal{R}_{\text{cont}} = \{x \in J^1(\Lambda^1 V) \ : \ bs(x) \wedge (Dx)^n \neq 0\}$$

(here $\dim X = 2n + 1$).

Surjectivity on π_0: Let F be a section of $J^1(\Lambda^1 V) \to \Lambda^1 V$ with image in $\mathcal{R}_{\text{cont}}$.

Fact 4.4 *The open manifold V contains a polyhedron $K \subset V$ of positive codimension such that V can be retracted by an isotopy $\phi_t : V \to V$, $t \in [0,1]$, into an arbitrarily small neighborhood of K.*

Indeed, triangulate V. Pick disjoint paths from the barycenter of each $(2n+1)$-simplex to ∞ (i.e. the path either ends in the boundary of V or exits every compact subset). Use these paths to isotope V into the complement of the barycenters of all $(2n+1)$-simplices. Finally, retract each punctured $(2n+1)$-simplex to a neighborhood of its boundary.

Now let $K \subset V$ be a polyhedron as in the fact above. Holonomic approximation gives

- a diffeotopy $h^t : V \to V$, $t \in [0,1]$
- a holonomic section $\tilde{F}$ of $J^1(\Lambda^1 V)$ on $\mathcal{O}p\ h^1 K$, C^0-close to $F|_{\mathcal{O}p\ h^1 K}$.

Notation 4.5 *Let $(\alpha, \beta) = (bs\ F, DF)$ and $(\tilde{\alpha}, d\tilde{\alpha}) = (bs\ \tilde{F}, D\tilde{F})$.*

By openness of $\mathcal{R}_{\text{cont}}$ and C^0-proximity, we can assume $(\tilde{\alpha}, d\tilde{\alpha})$ lies in $\mathcal{R}_{\text{cont}}$, as does the linear homotopy (over $\mathcal{O}p\ h^1(K)$) (α_t, β_t) from (α, β) to $(\tilde{\alpha}, d\tilde{\alpha})$.

Let $g_t : V \to V$ be an isotopy compressing V into $\mathcal{O}p\ h^1(K)$. Then

- $g_1^*(\tilde{\alpha}, d\tilde{\alpha})$ is a holonomic section of $J^1(\Lambda^1 V)$ with image in $\mathcal{R}_{\text{cont}}$
- the concatenation of the homotopies $g_t^*(\alpha, \beta)$ and $g_1^*(\alpha_t, \beta_t)$ gives a homotopy from (α, β) to $g_1^*(\tilde{\alpha}, d\tilde{\alpha})$ in $\mathcal{R}_{\text{cont}}$.

Finally, for injectivity on π_0 and for higher homotopy groups we simply mimick the above argument parametrically and apply the parametric holonomic approximation theorem. ∎

Remark 4.6 *Similar techniques prove an h-principle for symplectic forms (in a fixed cohomology class) on open manifolds.*

In fact, Gromov abstracted the main ideas in the above proof to prove the following general h-principle for open manifolds.

Theorem 4.7 *(Gromov) Let V be an open manifold and $X \to V$ a fiber bundle with a natural action of diff(V) on X. Let $\mathcal{R} \subset J^r(X)$ be an open, diff(V)-invariant subset. Then the inclusion*

$$hol\ sec\ \mathcal{R} \hookrightarrow sec\ \mathcal{R}$$

is a homotopy equivalence.

References

[EM] Y. Eliashberg and Nikolai M. Mishachev. *Introduction to the h-principle.* Number 48. American Mathematical Soc., 2002.