Homological Mirror Symmetry for $\mathbb{CP}^1$

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Goal: show that $\mathbb{CP}^1$ is mirror to $(\mathbb{C}^*, W(z) := z + 1/z)$. Here the pair $(\mathbb{C}^*, W(z))$ of a complex manifold and a holomorphic function on it is called a "Landau-Ginzburg model".

Remark 0.1 *$\mathbb{CP}^1$ is Fano, rather than Calabi-Yau, so its mirror is a Landau-Ginzburg model rather than just a manifold.*

Outline:

1. B-branes on $\mathbb{CP}^1$
2. A-branes on (C^*, W)
3. B-branes on (C^*, W)
4. A-branes on $\mathbb{CP}^1$

Mirror symmetry should give correspondences (1) $\leftrightarrow$ (2) and (3) $\leftrightarrow$ (4).

1 B-branes on $\mathbb{CP}^1$

Definition 1.1 *We take the category of B-branes on $\mathbb{CP}^1$ to be $D^b Coh(\mathbb{CP}^1)$.*

This is a strong invariant:

Theorem 1.2 *For X, Y smooth projective varieties with ω_X or ω_X^{-1} ample, $D^b Coh(X)$ equivalent to $D^b Coh(Y)$ implies that X is isomorphic to Y.*

Coherent sheaves on $\mathbb{CP}^1$ are classified:

Theorem 1.3 *Any coherent sheaf on $\mathbb{CP}^1$ is a finite direct sum of line bundles and skyscraper sheaves.*

Thus studying $D^b\text{Coh}(\mathbb{CP}^1)$ reduces to studying the coherent sheaf

$$\mathcal{E} := \mathcal{O} \oplus \mathcal{O}(1).$$

Lemma 1.4 *The smallest triangulated subcategory of $D^b Coh(\mathbb{CP}^1)$ containing $\mathcal{E}$ and closed under taking direct summands is $D^b Coh(\mathbb{CP}^1)$ itself.*

Lemma 1.5 $Hom(\mathcal{O}(m), \mathcal{O}(n)) \simeq \mathcal{O}(n-m)$.

Lemma 1.6 *Global sections of $\mathcal{O}(d)$ (on $\mathbb{CP}^1$) correspond to homogeneous degree d elements of $\mathbb{C}[x, y]$.*

Lemma 1.7 $Ext^i(\mathcal{E}, \mathcal{E}) = 0$ *for* $i > 0$.

Remark 1.8 $Ext^i_{Coh(\mathbb{CP}^1)}(X, Y) = Hom_{D^b Coh(\mathbb{CP}^1)}(X, Y[i])$.

So the only interesting Ext group of $\mathcal{E}$ is $\mathrm{Ext}^0(\mathcal{E}, \mathcal{E}) = \mathrm{Hom}(\mathcal{E}, \mathcal{E})$.

Let Q be the quiver $\cdot \rightrightarrows \cdot$, and let $\mathrm{PA}(Q)$ denote its path algebra.

Lemma 1.9 $Hom(\mathcal{E}, \mathcal{E}) \cong PA(Q)$.

Remark 1.10 *The two arrows correspond to the morphisms $x, y : \mathcal{O} \to \mathcal{O}(1)$.*

Proposition 1.11 *$D^b Coh(\mathbb{CP}^1)$ is triangle equivalent to $D^b(PA(Q){-}mod)$, where the latter denotes the derived category of modules over the algebra $PA(Q)$.*

2 A-branes on (C^*, W)

Equip $\mathbb{CP}^1$ with the symplectic form $\omega = \dfrac{idz \wedge d\overline{z}}{z\overline{z}}$. Observe that $W'(z) = 1 - 1/z^2$, hence the critical points of W occur when $z = \pm 1$. Moreover, $W''(z) = 2/z^3$, and $W''(\pm 1) = \pm 2$, hence W has non-degenerate (i.e. Lefshetz type) critical points.

Now pick a regular value $q \in C$. Then the regular fiber $W^{-1}(q) \subset \mathbb{C}^*$ consists of two points. Pick paths in $\mathbb{C}$ joining q to -2 and to 2. Then we get corresponding vanishing cycles in $W^{-1}(q)$ and vanishing thimbles $L_1, L_2 \subset \mathbb{C}$.

Definition 2.1

- *Let $FS(W)$ be the $\mathcal{A}_\infty$-category whose objects are a collection of vanishing thimbles L_i, with morphisms*

$$Hom(L_i, L_j) := \begin{cases} CF(\partial L_i, \partial L_j) & \text{if} \quad i < j \\ k \; id_{L_i} & \text{if} \quad i = j \\ 0 & \text{otherwise}, \end{cases}$$

  with the $\mathcal{A}_\infty$ structure maps on the vanishing cycles of $W^{-1}(q)$.

- *The category of A-branes is the bounded derived idempotent completion of the above, denoted $D^\pi FS(W)$.*

Remark 2.2

- $D^b FS(W) = H^0 \, TwFS(W)$.

- *A morphism $e : X \to X$ is* idempotent *if $e \circ e = e$, and* split *if there is an object Y and morphisms $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = e$ and $f \circ g = \mathbb{I}$.*

Remark 2.3 *We have seen earlier in the seminar that, up to equivalence, $D^\pi FS(W)$ is independent of the choice of vanishing paths.*

Claim: The morphism algebra of FS(W) is PA(Q), and $D^\pi\mathrm{FS}(W)$ is the bounded derived category of finite dimensional representations of Q.

Therefore $D^\pi\mathrm{FS}(W) \simeq D^b\mathrm{Coh}(\mathbb{CP}^1)$.

3 B-branes on (C^*, W)

Remark 3.1 *Coherent sheaves on $\mathbb{C}^*$ are equivalent to $\mathbb{C}[z, z^{-1}]$-modules, but we need to use W somehow.*

Definition 3.2
- *For A an algebra and $f \in A$, a* matrix factorization *of f consists of P_0, P_1 projective modules over A, and $d_P^0 : P_0 \to P_1$ and $d_P^1 : P_1 \to P_0$ such that $d_P^0 d_P^1 = d_P^1 d_P^0 = f\mathbb{I}$.*
- *For P^*, Q^* matrix factorizations, set*

$$Hom_{MF(f)}(P^*, Q^*) = Hom_A(P_0 \oplus P_1, Q_0 \oplus Q_1),$$

  $\mathbb{Z}/2$-graded with degree 0 piece: $Hom_A(P_0, Q_0) \oplus Hom_A(P_1, Q_1)$ and degree 1 piece: $Hom_A(P_0, Q_1) \oplus Hom_A(P_1, Q_0)$.
- *Equip $Hom_{MF(f)}(P^*, Q^*)$ with a differential:*

$$d_{P,Q}\phi = d_Q\phi \pm \phi d_P.$$

- *The category of B-branes of $(\mathbb{C}^*, W)$ is $\prod_{\lambda \in \mathbb{C}} MF(W - \lambda)$*

Fact: $d_{P,Q}^2 = 0$.

Definition 3.3 *MF(f) is the category with objects matrix factorizations and morphisms:*

$$H^0(Hom_{MF(f)}(P^*, Q^*)).$$

Proposition 3.4 *MF(f) is triangulated.*

Now consider the slightly more general $W(z) = z + Q/z$ (with $Q = 1$ in our case). Then W has critical points $\pm\sqrt{Q}$ with critical values $\pm 2\sqrt{Q}$, and $W \pm 2\sqrt{Q} = (1/z)(z \pm \sqrt{Q})^2$. Let $F_\pm$ be the matrix factorization for $W \mp 2\sqrt{Q} \in \mathbb{C}[z, z^{-1}]$, with $P_0 = P_1 = \mathbb{C}[z, z^{-1}]$, $d_P^0 = (1/z)(z \mp \sqrt{Q})$, and $d_P^1 = (z \mp \sqrt{Q})$.

Proposition 3.5 *$Hom_{MF(W\mp 2\sqrt{Q})}(F_\pm, F_\pm)$ is isomorphic to the Clifford algebra on a single generator ϕ, with $\phi \cdot \phi = -(\partial_z^2 W)(\pm\sqrt{Q})$.*

Claim:

- 0 and $F_\pm$ generate $MF(W \mp 2\sqrt{Q})$
- The category of B-branes for $(\mathbb{C}^*, z + Q/z)$ is equivalent to

$$D^b(k\langle x\rangle/\langle x^2 = 1/\sqrt{Q}\rangle) \times D^b(k\langle x\rangle/\langle x^2 = -1/\sqrt{Q}\rangle)$$

(i.e. $\mathbb{C}[x]/\langle x^2 = \pm 1\rangle$ in the simplest case).

4 A-branes on $\mathbb{CP}^1$

- We take the category of A-branes on $\mathbb{CP}^1$ to be $D^\pi(\mathbb{CP}^1)$.
- The Fukaya category of $\mathbb{CP}^1$ is indexed by "charge" $\lambda \in \mathbb{C}$, where Fuk$(\mathbb{CP}^1, \lambda)$ consists of *weakly unobstructed Lagrangians* with $m_0 = \lambda \cdot [L]$. Here weakly unobstructed means μ_L^0 is a scalar multiple (the "central charge" or "superpotential") of the cohomological unit of Hom(L, L).
- We should really take pairs (L, ∇), where ∇ is a flat unitary connection on a vector bundle.
- $HF(L^\nabla, L^\nabla) = 0$ unless L is the equator and hol$(\nabla) = \pm\mathbb{I}$.
- $HF(L^\nabla, L^\nabla)$ in this case is isomorphic to $H^*(S^1, \mathbb{C})$ as a module, but with multiplicative structure making it isomorphic to $\mathbb{C}[x]/\langle x^2 = \pm 1\rangle$.

References

[Aur] Denis Auroux. http://www-math.mit.edu/ auroux/18.969/, Lectures 24 and 25. (2009).

[Bal] Matthew Robert Ballard. Meet homological mirror symmetry. *arXiv preprint arXiv:0801.2014* (2008).