Exterior Differential Systems*

Kyler Siegel

February 23, 2014

1 Exterior Differential Systems and Basic Examples

Definition 1.1 *An* exterior differential system *(EDS) on a smooth manifold M consists of a graded differential ideal $\mathcal{I}$ of the ring $\Omega^*(M)$ of differential forms on M. In particular, $\mathcal{I}$ being differential means it is closed under the exterior derivative $d : \Omega^p(M) \to \Omega^{p+1}(M)$, while graded means it decomposes as*

$$\mathcal{I} = \bigoplus_{p=0}^{\infty} \mathcal{I}^p,$$

where $\mathcal{I}^p = \mathcal{I} \cap \Omega^p(M)$.

We will denote by $\langle \gamma_1, ..., \gamma_k \rangle$ the *differential* ideal generated by $\gamma_1, ..., \gamma_k$, i.e. the set of elements of $\Omega^*(M)$ of the form

$$\alpha^1 \wedge \gamma_1 + ... + \alpha^k \wedge \gamma_k + \beta^1 \wedge d\gamma_1 + ... + \beta^k \wedge d\gamma_k,$$

for some $\alpha^1, ..., \alpha^k, \beta^1, ..., \beta^k \in \Omega^*(M)$. We also denote by $\langle \gamma_1, ..., \gamma_k \rangle_{\text{alg}}$ the "algebraic" ideal (not necessarily closed under d) consisting of elements of the form

$$\alpha^1 \wedge \gamma_1 + ... + \alpha^k \wedge \gamma_k.$$

The basic problem of EDS is to find *integral submanifolds* of M, i.e. submanifolds $\iota : N \hookrightarrow M$ such that $\iota^*\mathcal{I} = (0)$.

Example 1.2 *Consider the system of ordinary differential equations*

$$y'(x) = (xyz)^{17}$$
$$z'(x) = \cosh(x + y + z).$$

*Notes for a talk given on 2/20/14 at Stanford University. Our primary reference is [Bry]

We can model this system by an EDS with $M = \mathbb{R}^3$ and

$$\mathcal{I} = \langle dy - (xyz)^{17}dx, dz - cosh(x + y + z)dx\rangle.$$

Note that the 1-dimensional integral manifolds of $\mathcal{I}$ are precisely the integral curves of the vector field

$$\frac{\partial}{\partial x} + (xyz)^{17}\frac{\partial}{\partial y} + \cosh(x + y + z)\frac{\partial}{\partial z}.$$

Example 1.3 *Consider of the system of partial differential equations given by*

$$\frac{\partial u}{\partial x}(x, y) = F(x, y, u(x, y))$$
$$\frac{\partial u}{\partial y}(x, y) = F(x, y, u(x, y)).$$

We can model this by an EDS in $\mathbb{R}^3$, with

$$\mathcal{I} = \langle dz - F(x, y, z)dx - G(x, y, z)dy\rangle.$$

Note that a surface $N \subset M$ is integral if and only if both of the vector fields

$$\frac{\partial}{\partial x} + F(x, y, z)\frac{\partial}{\partial z} \quad \text{and} \quad \frac{\partial}{\partial y} + G(x, y, z)\frac{\partial}{\partial z}$$

are tangent to N. Of course, there need not be any 2-dimensional integral manifolds, for example if $F(x, y, z) = y$ and $G(x, y, z) = -x$.

Example 1.4 *Complex curves in $\mathbb{C}^2$ Let $M = \mathbb{C}^2$ with coordinates $z = x+iy$ and $w = u+iv$, and let*

$$\mathcal{I} = \langle Re(dz \wedge dw), Im(dz \wedge dw)\rangle = \langle dx \wedge du - dy \wedge dv, dx \wedge dv + dy \wedge du\rangle.$$

In this case any real curve in $\mathbb{C}^2$ is integral, since $\mathcal{I}^1 = (0)$. On the other hand, a real surface $N \subset \mathbb{C}^2$ is integral if and only if it is a complex curve.

Note that N can be written locally as a graph $\{(z, u(z)+iv(z)\}$, with $u+iv$ a holomorphic function of z, whenever dx and dy are linearly independent on N. The condition $dx \wedge dy \neq 0$ on N is sometimes called an independence condition. *Note that $u + iv$ being holomorphic is equivalent to*

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0,$$

i.e. $\mathcal{I}$ (with the independence condition) is a model for the Cauchy-Riemann equations.

Example 1.5 *Consider a system of first order partial differential equations of the form*

$$F^i(\mathbf{x}, \mathbf{z}, \frac{\partial \mathbf{z}}{\partial \mathbf{x}}) = 0, \qquad i = 1, ..., r,$$

for $\mathbf{x} = (x^1, ..., x^a)$ *independent variables,* $\mathbf{z} = (z^1, ..., z^b)$ *dependent variables, and* $\frac{\partial \mathbf{z}}{\partial \mathbf{x}}$ *the Jacobian matrix of* $\mathbf{z}$ *with respect to* $\mathbf{x}$*. Assume that the common zero set* $M^{a+b+ab-r}$ *of* $\{F^i\}$ *in* $\mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^{ab}$ *(with coordinates* $(\mathbf{x}, \mathbf{z}, \mathbf{y})$*) is cut out transversely, i.e. that* $(F^1, ..., F^r) : \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^{ab} \to \mathbb{R}^r$ *is a submersion. We define an EDS on* M *by*

$$\langle dz^1 - \sum_{i=1}^{a} y_i^1 dx^i, ..., dz^b - \sum_{i=1}^{a} y_i^b dx^i \rangle.$$

Then the n*-dimensional integral submanifolds* $N \subset M$ *of* $\mathcal{I}$*, satisfying the independence condition* $dx^1 \wedge ... \wedge dx^a \neq 0$*, are precisely the graphs of solutions of the original PDE system.*

Example 1.6 *We can certainly use this same principle to encode higher order PDE systems as well. For example, an equation of the form*

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0,$$

with $M^7 = \{F(x, y, u, p, q, r, s, t) = 0\} \subset \mathbb{R}^8$ *a smooth hypersurface, can be encoded using*

$$\mathcal{I} = \langle du - pdx - qdy, dp - rdx - sdy, dq - sdx - tdy \rangle.$$

2 The Frobenius Theorem

In what follows, we call an EDS $(M, \mathcal{I})$ *Frobenius* if $\mathcal{I} = \langle \mathcal{I}^1 \rangle_{\text{alg}}$ with $\dim \mathcal{I}^1_p = r$ constant as p varies.

Theorem 2.1 *Let* $(M^m, \mathcal{I})$ *be an EDS which is Frobenious. Then each* $p \in M$ *has a local coordinate chart* $(y^1, ..., y^m)$ *on which*

$$\mathcal{I} = \langle dy^1, ..., dy^{m-n} \rangle.$$

In particular, the local n*-dimensional integral manifolds of* $\mathcal{I}$ *are just slices*

$$\{y^1 = c^1, ..., y^{m-n} = c^{m-n}\}.$$

Example 2.2 *Recall that the system*

$$\frac{\partial u}{\partial x}(x, y) = F(x, y, u(x, y))$$
$$\frac{\partial u}{\partial y}(x, y) = F(x, y, u(x, y)).$$

became the EDS with $\mathcal{I} = \langle \zeta \rangle$, where

$$\zeta = dz - F(x,y,z)dx - G(x,y,z)dy.$$

Now the condition for the Frobenius theorem is that $\langle \zeta \rangle = \langle \zeta \rangle_{alg}$, which by simple linear algebra is equivalent to

$$0 = \zeta \wedge d\zeta = (F_y - G_x + GF_z + FG_z)dx \wedge dy \wedge dz.$$

Hence, if F and G satisfy the auxiliary equation

$$F_y - G_x + GF_z - FG_z = 0,$$

then the Frobenius Theorem gives for any $(x_0, y_0, z_0) \in \mathbb{R}^3$ a function $u(x,y)$ defined near (x_0, y_0) with

$$\begin{aligned} u(x_0, y_0) &= z_0 \\ u_x &= F(x, y, u(x,y)) \\ u_y &= G(x, y, u(x,y)). \end{aligned}$$

Remark 2.3 *For α a 1-form on an odd-dimensional manifold M^{2n+1}, we have seen that $\alpha \wedge d\alpha = 0$ implies the hyperplane distribution* $\ker \alpha$ *gives a foliation on M. At the other extreme, if $\alpha \wedge d\alpha \wedge ... \wedge d\alpha \neq 0 \in \Omega^{2n+1}(M)$, the hyperplane distribution* $\ker \alpha$ *is called a* contact structure*. In many ways, contact structures behave like symplectic structures on even dimensional manifolds.*

Remark 2.4 *Before proving the Frobenius theorem, we would like to relate our version to the perhaps more familiar version about distributions in the tangent plane of a manifold M. Namely, if M has an n-plane distribution D which is* involutive*, then M admits local coordinate charts near any point in which D is spanned by the first n coordinate vector fields. Here involutivity means that the Lie bracket of any two (locally defined) vector fields tangent to D is again tangent to D. We show that this version is equivalent to the version stated at the beginning of this section.*

Firstly, assume $(M, \mathcal{I})$ is Frobenius, and let D be the distribution consisting of tangent vectors on which $\mathcal{I}^1$ vanishes. For $\theta \in \mathcal{I}^1$ and X, Y local vector fields tangent to D, we have

$$d\theta(X,Y) = X\theta(Y) - Y\theta(X) - \theta([X,Y]),$$

with $\theta(X)$ and $\theta(Y)$ vanishing since X and Y are tangent to D. Since $d\theta(X,Y) \in \langle \mathcal{I}^1 \rangle_{alg}$, we also have $d\theta(X,Y) = 0$. Therefore $\theta([X,Y]) = 0$, so $[X,Y]$ is also tangent to D.

Now assume that D is an involutive n-plane distribution, and let $\mathcal{I}$ be the differential ideal generated by 1-forms vanishing on D. We wish to show that $\mathcal{I}$ is Frobenius, i.e. that $d\theta \in \mathcal{I} \in \langle \mathcal{I}^1 \rangle_{alg}$ for any $\theta \in \mathcal{I}^1$. Let $e_1, ..., e_{n+r}$ be a local frame for TM, with $e_1, ..., e_n$ tangent to D, and let $\theta^1, ..., \theta^{n+r}$ be the dual frame for T^*M. Then we can expand $d\theta$ as

$$d\theta = \sum_{1 \leq i < j \leq n} A_{ij}\theta^i \wedge \theta^j + \sum_{1 \leq i \leq n,\, n+1 \leq j \leq n+r} B_{ij}\theta^i \wedge \theta^j + \sum_{n+1 \leq i < j \leq n+r} C_{ij}\theta^i \wedge \theta^j.$$

Note that the formula for $d\theta$ in the previous paragraph shows that $d\theta(X,Y)$ whenever X and Y are tangent to D, and so in particular $d\theta(e_i, e_j) = 0$ whenever $1 \leq i,j \leq n$. Therefore we have $A_{ij} = 0$ for all ij. Since $\theta^j \in \mathcal{I}^1$ whenever $n+1 \leq j \leq n+r$, we have $d\theta$ in the form $\sum_i \beta_i \wedge \gamma_i$ for $\gamma_i \in \mathcal{I}^1$, hence $d\theta \in d\theta \in \mathcal{I} \in \langle \mathcal{I}^1 \rangle_{alg}$.

Proof (Frobenius theorem) Assume $\mathcal{I}$ is generated by $m-n$ linearly independent 1-forms $\theta^1, ..., \theta^{m-n}$, and let D be the distribution on which the θ^i vanish. Following [BGC+], we proceed by induction on n. Firstly, if $n=1$ then D is line field and we can find "flowbox coordinates". Now assume the theorem is true up to $n-1$.

We start by picking $x : M \to \mathbb{R}$ a smooth function such that $\theta^1 \wedge ... \wedge \theta^{m-n} \wedge dx \neq 0$ near the point p. Then $\langle \theta^1, ..., \theta^{m-n}, dx \rangle$ is Frobenius, and so by the induction hypothesis we can find local coordinates $(y^1, ..., y^m)$ with

$$\langle dy^1, ..., dy^{m-n-1} \rangle = \langle \theta^1, ..., \theta^{m-n}, dx \rangle.$$

Then we can write

$$dx = \sum_{i=1}^{n-m} a_i dy^i + a_{m-n+1} dy^{m-n+1},$$

$$\theta^i = \sum_{j=1}^{n-m} c^i_j dy^j + c^i_{m-n+1} dy^{m-n+1}, \qquad 1 \leq i \leq n-m,$$

for a_i, c^i_j smooth functions. Assuming without loss of generality that $a_{m-n+1} \neq 0$, we can solve for dy^{m-n+1} in the first equation and use this to rewrite the second equation as

$$\theta^i = \sum_{j=1}^{n-m} \tilde{c}^i_j dy^j + f^i dx$$

for smooth functions $\tilde{c}^i_j, f^i$. Observe that the matrix of functions $\tilde{c}^i_j$ is invertible near p (since otherwise we could find a nontrivial dependency between the θ^i and dx). Therefore we can write $\mathcal{I} = \langle \tilde{\theta^1}, ..., \theta^{\tilde{n-m}} \rangle$, where

$$\tilde{\theta}^i = dy^i + e^i dx$$

for e^i smooth functions and $\theta^i = \sum_{j=1}^{n-m} \tilde{c}^i_j \tilde{\theta}^j$. Therefore we have $d\tilde{\theta}^i = de^i \wedge dx$, and hence

$$de^i \wedge dx \equiv 0 \bmod \{\tilde{\theta}^i\}.$$

Therefore we can write

$$de^i = a dx + \sum_{j=1}^{n-m} b^i_j dy^j$$

for some functions a, b^i_j. But now we observe that dx and each de^i have expansions involving $dy^1, ..., dy^{m-n+1}$ but not $dy^{m-n+2}, ..., dy^m$, and therefore x and each e^i are functions of $y^1, ..., y^{m-n+1}$ only. Therefore the same is true of the $\tilde{\theta}^i$.

Now let V be the submanifold through p obtained by setting $y^{m-n+2}, ..., y^m$ to be constant. Observe that $\mathcal{I}|_V$ is a codimension one Frobenius system, and hence we can found coordinates $(\tilde{y}^1, ..., \tilde{y}^{m-n+1})$ on V such that $\mathcal{I}|_V$ is generated by $d\tilde{y}^1, ..., d\tilde{y}^{m-n}$. Then we can take

$$(\tilde{y}^1, ..., \tilde{y}^{m-n+1}, y^{m-n+2}, ..., y^m)$$

as the promised coordinate system.

3 The Pfaff and Darboux Theorems

Exterior differential systems generated by a single one-form are called *Pfaffian* systems. Like Frobenius systems, these also have nice local normal forms.

Theorem 3.1 *(Pfaff) Let α be a one-form on a manifold M^n. Assume that in a neighborhood the number r (called the* rank *of α) defined by*

$$\alpha \wedge (d\alpha)^r \neq 0, \qquad \alpha \wedge (d\alpha)^{r+1} = 0$$

is constant. Then M has local coordinates $(w^1, ..., w^n)$ in which $\mathcal{I} = \langle \alpha \rangle$ becomes

$$\langle dw^1 + w^2 dw^3 + ... + w^{2r} dw^{2r+1} = 0.$$

In fact, under a slightly stronger assumption there is a normal form for the contact form α itself, and not just its kernel.

Theorem 3.2 *Let α be a one-form on a manifold M^n. Assume that in a neighborhood the numbers r and s defined by*

$$\alpha \wedge (d\alpha)^r \neq 0, \qquad \alpha \wedge (d\alpha)^{r+1} = 0$$

and

$$(d\alpha)^s \neq 0, \qquad (d\alpha)^{s+1} = 0$$

are constant. Then M has local coordinates $(y^1, ..., y^n)$ in which α is given by

$$\alpha = y^0 dy^1 + ... + y^{2r} dy^{2r+1}, \qquad \text{if } r + 1 = s$$
$$\alpha = dy^1 + y^2 dy^3 + ... + y^{2r} dy^{2r+1}, \qquad \text{if } r = s.$$

There is also a similar local normal form for *closed* two-forms.

Theorem 3.3 *(Darboux) Let Ω be a closed two-form on a manifold M such that the number r defined by*

$$\Omega^r \neq 0, \qquad \Omega^{r+1} = 0$$

is constant. Then M has local coordinates $(w^1, ..., w^n)$ in which Ω is given by

$$\Omega = dw^1 \wedge dw^2 + ... + dw^{2r-1} \wedge dw^{2r}.$$

Example 3.4 *Consider a single first order PDE of the form*

$$F(x^1, ..., x^n, u, \frac{\partial u}{\partial x^1}, ..., \frac{\partial u}{\partial x^n}) = 0.$$

Assuming F is nice enough, it cuts out a hypersurface

$$M = \{(x^1, ..., x^n, u, p_1, ..., p_n) \mid F(x^1, ..., x^n, u, p_1, ..., p_n) = 0\} \subset J^1(\mathbb{R}^n, \mathbb{R}).$$

For $\theta = du - p_1 dx^1 - ... - p_n dx^n$, the PDE is encoded by the EDS

$$\mathcal{I} = \langle \theta \rangle.$$

We have $\theta \wedge (d\theta)^n = 0$, while $\theta \wedge (d\theta)^{n-1}$ is nowhere vanishing, and therefore by the Pfaff theorem we can find local coordinates $(z, y^1, ..., y^{n-1}, v, q_1, ..., q_{n-1})$ on M on which $\mathcal{I}$ is given by

$$\langle dv - q_1 dy^1 - q_2 dy^2 - ... - q_{n-1} dy^{n-1} \rangle.$$

Now we observe that an n-dimensional integral manifold of $\mathcal{I}$ is locally of the form

$$v = g(y^1, ..., y^{n-1}), \qquad q_i = \frac{\partial g}{\partial y_i}(y^1, ..., y^{n-1})$$

for some function $g : \mathbb{R}^{n-1} \to \mathbb{R}$. In particular, it is tangent to the vector field $Z = \dfrac{\partial}{\partial z}$.

Remark 3.5 *One can check that, up to a multiple, Z is given by*

$$\sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \frac{\partial}{\partial x^i} + \sum_{i=1}^{n} p_i \frac{\partial F}{\partial p_i} \frac{\partial}{\partial u} - \sum_{i=1}^{n} \left(\frac{\partial F}{\partial x^i} + p_i \frac{\partial F}{\partial u} \right) \frac{\partial}{\partial p_i}$$

This is called the Cauchy characteristic vector field *of F, and this method of solving the PDE in a direction transverse to Z and then "thickening up" by the integral curves of Z is known as the* method of characteristics.

4 The Cartan-Kahler Theorem

We now want to state (a baby version of) the Cartan-Kahler Theorem. First, some terminology is in order. An *integral element* of an EDS $(M, \mathcal{I})$ is an n-dimensional subspace $E \subset T_xM$ such that

$$\phi(v_1, ..., v_n) = 0$$

for all $\phi \in \mathcal{I}^n$ and all $v_1, ..., v_n \in E$. Let $V_n(\mathcal{I}) \subset G_n(TM)$ denote the set of all n-dimensional integral elements of $\mathcal{I}$. We say $EinV_n(\mathcal{I})$ is an *ordinary element* if $V_n(\mathcal{I}) \subset G_n(TM)$ is cut out transversely near E (this can be made more precise using the natural local coordinates on $G_n(TM)$).

We also define the *polar space* $H(E)$ of $E \in V_n(\mathcal{I})$ by

$$H(E) = \{v \in T_xM \mid \kappa(v, e_1, ..., e_k) = 0 \forall \kappa \in \mathcal{I}^{k+1}\} \subset T_xM.$$

The polar space measures the possible extensions of E to a larger integral element. We define a function which measures the number of possible extensions by

$$r : V_k(\mathcal{I}) \to \{-1, 0, 1, 2, ...\}$$
$$r(E) = \dim H(E) - k - 1.$$

So $r(E) = -1$ means there are no possible extensions, $r(E) = 0$ means there is a unique extension, and so on. We say $E \in V_n(\mathcal{I})$ is *regular* if r is constant in a neighborhood of E in $V_n^0(\mathcal{I})$.

The following theorem is not the most general form of Cartan-Kahler, but will suffice for this talk.

Theorem 4.1 *(Cartan-Kahler) Let $(M, \mathcal{I})$ be a real analytic EDS. If $E \in V_n(\mathcal{I})$ contains a flag of subspaces*

$$(0) = E_0 \subset E_1 \subset ... \subset E_{n-1} \subset E_n = E \subset T_pM$$

with $E_i \in V_i^r(\mathcal{I})$ for $0 \leq i < n$, then there is a real analytic n-dimensional integral manifold passing through p and tangent to E at p.

The reason for the flag is that the proof works by inductively extending an integral manifold to a larger one of dimension one greater. We call a flag $E_0, ..., E_n$ as above a *regular flag.* Notice that we do not require E_n to be a regular element.

The hypotheses of this theorem look rather difficult to check, but fortunately there is a simple test. We consider the sequence of numbers

$$0 \leq c(E_0) \leq c(E_1) \leq ... \leq c(E_n)$$

with $c(E_k)$ defined to be the codimension of the polar space of E_k:

$$c(E_k) = \dim(T_pM) - \dim H(E_k).$$

Theorem 4.2 *(Cartan's Test) Let* $(M, \mathcal{I})$ *be an EDS and* $E_0, E_1, ..., E_n$ *an intgral flag of* $\mathcal{I}$. *Then* $V_n(\mathcal{I})$ *has codimension at least*

$$c(F) = c(E_0) + c(E_1) + ... + c(E_{n-1})$$

in $G_n(TM)$ *at* E_n. *Moreover,* $V_n(\mathcal{I})$ *is actually a smooth submanifold of* $G_n(TM)$ *of codimension* $c(F)$ *near* E_n *if and only if the flag* F *is regular.*

Example 4.3 *Let's see what the Cartan-Kahler theorem says for a Frobenius system* $(M^{n+s}, \mathcal{I})$, *where* $\mathcal{I}$ *is generated algebraically by linearly independent* 1*-forms* $\theta^1, ..., \theta^s$. *Pick an integral element* $E \in V_n(\mathcal{I})$, *i.e.* $E \subset T_pM$ *is* n *dimensional and* $\theta^1, ..., \theta^s$ *vanish on* E. *Pick any flag of subspaces*

$$(0) = E_0 \subset E_1 \subset ... \subset E_{n-1} \subset E_n = E.$$

Then each polar space $h(E_i)$ *is just* E *itself, hence each* $c(E_i) = (n+s) - n = s$. *Thus* $c(E_0, E_1, ..., E_n) = ns$. *On the other hand, since* $V_n(\mathcal{I})$ *consists of one element in each fiber of* $G_n(TM)$, *it has codimension* ns. *Hence Cartan's test tells us that the flag is regular, and therefore there is an integral* n*-manifold tangent to* E. *Of course, we already knew this from the Frobenius theorem.*

Example 4.4 *Consider the EDS* $(M^{2n+1}, \mathcal{I} = \langle \alpha \rangle)$, *where* α *is a contact* 1*-form. Then* $\ker \alpha$ *defines a (non-integrable) hyperplane distribution on* M, *and an integral element* E *must be* isotropic *with respect to* $d\alpha$. *Therefore the integral elements have dimension at most* n, *and the* n*-dimensional integral elements in* T_pM *are the Lagrangians of the symplectic vector space* $(\ker \alpha_p, d\alpha|_{\ker \alpha_p})$. *Let* E *be such a Lagrangian, and pick an flag*

$$(0) = E_0 \subset E_1 \subset ... \subset E_n = E.$$

Then one easily checks that we have

$$\begin{aligned} c(E_o) &= (2n+1) - (2n) = 1 \\ c(E_1) &= (2n+1) - (2n-1) = 2 \\ &... \\ c(E_{n-1}) &= (2n+1) - (n+1) = n, \end{aligned}$$

and therefore

$$c(E_0, E_1, ..., E_n) = \frac{n(n+1)}{2}.$$

On the other hand, the Lagrangian Grassmanian of $(\ker \alpha_p, d\alpha|_{\ker \alpha_p})$ *has dimension* $\frac{n(n+1)}{2}$, *hence* $V_n(\mathcal{I})$ *has codimension* $\frac{n(n+1)}{2}$. *Therefore we can apply Carta's test applies. In contact geometric lingo, Cartan-Kahler says that we can find a Legendrian of* $(M, \ker \alpha)$ *which is tangent to* E.

References

[Bry] Robert Bryant. Nine Lectures on Exterior Differential Systems, July 1999.

[BGC+] Robert L Bryant, PA Griffiths, SS Chern, Robert B Gardner, and Hubert L Goldschmidt. *Exterior differential systems*. Springer, 1991.

[IL] Thomas Andrew Ivey and Joseph M Landsberg. Cartan for beginners: differential geometry via moving frames and exterior differential systems. (2003).