DEFORMATIONS, QUANTIZATIONS, AND NONCOMMUTATIVITY

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1. Introduction

Classical mechanics can be viewed as the geometry of Poisson manifolds. We begin by recalling their definition.

Definition 1.1. A Poisson algebra is an associative algebra A over a field $\mathbb{K}$ (fixed, of characteristic zero), equipped with a Lie bracket $\{-,-\}$ such that $\{x,-\}$ is a derivation for any $x \in A$, i.e. $\{x, yz\} = \{x, y\}z + y\{x, z\}$.

Definition 1.2. A Poisson structure on a manifold M is a Poisson bracket $\{-,-\}$ on the algebra $C^\infty(M)$.

Example 1.3. On $T^*\mathbb{R}^n$ with position coordinates $q_1, ..., q_n$ and momentum coordinates $p_1, ..., p_n$, the standard bracket is given by

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right)$$

Example 1.4. Any symplectic manifold has a natural Poisson structure.

A Poisson structure determines the time evolution of the system. Namely, if for any "observable" $f : M \to \mathbb{R}$, we have $df/dt = \{H, f\}$, where H is the Hamiltonian of the system (for example of the form $H(q, p) = p^2/2m + V(q)$). As a sanity check, we have $\{H, H\} = 0$, i.e. conservation of energy.

In quantum mechanics, observables are instead given by Hermitian operators on some Hilbert space. Viewing the algebra of observables as the primary detail, we can formally pass from classical to quantum mechanics by defining a new associative product $*_h$ on $C^\infty(M)$ such that:

(1) $\lim_{h\to 0} f *_h g = fg$
(2) $\lim_{h\to 0} \dfrac{f *_h g - g *_h f}{2h} = \{f, g\}$.

The first condition represents the fact that quantum mechanics reduces to classical mechanics in the small h limit, whereas the second condition says that the commutator with respect to $*_h$ agrees with the Poisson bracket to first order. Said slightly differently, $(C^\infty(M), *_h)$ is a "deformation of $(C^\infty(M), \cdot)$ in the direction of $\{-,-\}$".

Question 1.5. *Does $*_h$ actually exist for a given Poisson manifold? If so, can we classify all possible $*_h$?*

2. Deformations of associative algebras

Let A be a finite dimensional associative algebra over $\mathbb{K}$. Consider a new product $a * b = ab + hf(a,b)$, for some $f \in \hom(A \otimes A, A)$. Here we view $*$ as a product on $A \otimes \mathbb{K}[h]/(h^2)$, i.e. we are studying the "first order deformation theory" of A.

Exerise 2.1. In terms of f, the associativity condition $(a * b) * c = a * (b * c)$ is equivalent to

$$f(ab, c) + f(a, b)c = f(a, bc) + af(b, c). \tag{2.1}$$

However, certain deformations should be considered equivalent. Namely, for T any linear automorphism of $A \otimes \mathbb{K}[h]/(h^2)$ of the form $T(a) = a + hg(a)$, for $g \in \hom(A, A)$, we put $* \sim *_T$, where

$$a *_T b := T(T^{-1}(a) * T^{-1}(b). \tag{2.2}$$

In other words, two deformations are equivalent if one can be conjugated to the other by a linear automorphism which is trivial modulo h.

Exerise 2.2. For $a * b = ab + hf(a, b)$, we can write $a *_T b = ab + hf_T(a, b)$, with

$$f_T(a, b) := f(a, b) - g(a)b - ag(b) + g(ab). \tag{2.3}$$

In summary, first order deformations of A up equivalence are in one-to-one correspondence with $f \in \hom(A \otimes A, A)$ satisfying (2.2), modulo (2.3) for any $g \in \hom(A, A)$. The latter is also known as $HH^2(A, A)$, the degree two Hochschild cohomology of A.

3. Hochschild cohomology and first order deformations

Define the Hochschild complex $CC^*(A, A)$ is of the form

$$\hom(\mathbb{K}, A) \to \hom(A, A) \to \hom(A \otimes A, A) \to \hom(A \otimes A \otimes A, A) \to ...,$$

where $d : \hom(A^{\otimes(n-1)}, A) \to \hom(A^{\otimes n}, A)$ is given by

$$(df)(a_1, ..., a_n) = a_1 f(a_2, ..., a_n) + \sum_{i=1}^{n-1} (-1)^i f(a_1, ..., a_i a_{i+1}, ..., a_n) + (-1)^n f(a_1, ..., a_{n-1})a_n.$$

The Hochschild cohomology is then $HH^i(A, A) := \ker d|_{\hom(A^{\otimes i}, A)}/\text{im } d$.

Example 3.1. Consider the associative algebra $A = \mathbb{K}[x]/(x^2 - 1)$, i.e. the group ring of $\mathbb{Z}/2$. We claim that $HH^2(A, A) = 0$, meaning there are no deformations of A, at least up to first order. This perhaps reflects the rigidity of finite groups.

Example 3.2. Now consider $A = \mathbb{K}[x]/(x^2)$. We claim that $HH^2(A, A)$ is one dimensional, with a nontrivial cocycle given by $f(x, x) = h$, $f(x, 1) = f(1, x) = f(1, 1) = 0$. In other words, A deforms to $\mathbb{K}[x]/(x^2 - h)$, which is essentially the group ring from the previous example.

4. The Hochschild DGLA

It turns out that the Hochschild complex $CC^*(A,A)$ also has a natural Lie bracket. Namely, for $f \in \hom(A^{\otimes n}, A)$ and $g \in \hom(A^{\otimes m}, A)$, we define $[f,g] \in \hom(A^{\otimes(n+m-1)}, A)$ by

$$[f,g](a_1,...,a_{n+m-1}) = \sum_{i=1}^{n}(-1)^{im} f(a_1,...,a_{i-1}, g(a_i,...a_{i+m-1}), a_{i+m},...,a_{n+m-1})$$
$$+ \sum_{j=1}^{m}(-1)^{jn} g(a_1,...,a_{j-1}, f(a_j,...a_{i+n-1}), a_{j+n},...,a_{n+m-1}).$$

Definition 4.1. A differential graded Lie algebra (DGLA) is a graded vector spae L, together with $[-,-]$ and d, where

- $[-,-]$ is graded antisymmetric and satisfies a graded Jacobi identity
- d is a differential and satisfies the graded Leibniz rule: $d[x,y] = [dx,y] + (-1)^{|x|}[x,dy]$.

It turns out that $CC^*(A,A)$ is in fact a DGLA. Now consider a deformed product of the form $a * b = ab + f(a,b)$. We claim that $*$ is associative if and only if $df + \frac{1}{2}[f,f] = 0$, i.e. f satisfies the "Maurer–Cartan equation" in $CC^*(A,A)$. As for equivalences of deformations, consider any linear automorphism $T : A \to A$. As before, we set $* \sim *_T$ for $a *_T b := T(T^{-1}(a) * T^{-1}(b))$. Since the Lie algebra of the general group of A is $\hom(A,A)$, we can uniquely write $T = e^g$ for $g \in \hom(A,A)$. Claim: for $a * b = ab + f(a,b)$, we have $a *_T b = ab + f_T(a,b)$, with f_T given by

$$f_T = e^g f e^{-g} + e^g d e^{-g}. \tag{4.1}$$

In this situation, one says that f and f_T are "gauge equivalent".

Remark 4.2. In any "nice" DGLA, we can define $\mathcal{M}$ to be the set of Maurer–Cartan elements modulo gauge equivalence. Here a Martan–Cartan element is by definition an element x of degree one such that $dx + \frac{1}{2}[x,x] = 0$. The gauge group is the Lie group associated to the Lie algebra formed by all elements of degree zero, and it naturally acts on Maurer–Cartan elements as in (4.1).

5. Obstructions to deformations

Question 5.1. *Suppose f_1 is a first order deformation of the associative algebra A, i.e. $a * b = ab + hf_1(a,b)$ is an associative product on $A \otimes \mathbb{K}[h]/(h^2)$. Can we extend this to a "formal deformation", i.e. a product $a * b = ab + \widetilde{f}(a,b)$ on $\mathbb{K}[[h]]$, where $\widetilde{f} = hf_1 + h^2 f_2 + h^3 f_3 + ...$?*

One can check that associativity of the product $a * b = ab + \widetilde{f}(a,b)$ is equivalent to the equation $d\widetilde{f} + \frac{1}{2}[\widetilde{f}, \widetilde{f}] = 0$, which is equivalent to the system of equations

$$df_n + \tfrac{1}{2}\sum_{i=1}^{n-1}[f_i, f_{n-i}] = 0 \text{ for all } n. \tag{5.1}$$

Moreover, one can check that $\frac{1}{2}\sum_{i=1}^{n-1}[f_i, f_{n-i}]$ is closed with respect to the Hochschild differential, and we are wondering if it is exact. In particular, it follows that vanishing of $HH^3(A, A)$ is a sufficient condition to be able to solve the system (5.1), and in this case we say that the deformation theory of A is "unobstructed".

6. Examples of deformation problems

There is a metaprinciple observed by Deligne is that every deformation problem (over a field of characteristic zero) is governed by some DGLA.

Example 6.1. If L is a finite dimensional Lie algebra, there is a story closely resembling the above one for associative algebras, except with the Hochschild complex replaced by the "Chevalley complex". The Chevalley complex of a Lie algebra is again a DGLA and its cohomology is the Lie algebra cohomology of L, which in nice cases is just the cohomology of the Lie group $\exp(L)$.

Example 6.2. Similarly, for the deformation theory of commutative algebras, one replaces the Hochschild complex with the "Harrison complex".

Example 6.3. Let X be a complex manifold with holomorphic tangent bundle T_X. "Nearby" complex structures on X up to equivalence correspond to Maurer–Cartan elements modulo gauge equivalence in the DGLA given by $\Gamma^* = \Gamma(X, \Omega_X^{0,*}(T_X))$. Here any element of Γ^* can be written locally as $F(z_1, ..., z_n)\partial_{z_i} \otimes d\overline{z_{i_1}} \wedge .... \wedge d\overline{z_{i_j}}$, and the differential is $\overline{\partial}$.

7. quantum groups

Definition 7.1. A Hopf algebra is an associative algebra A, equipped with a coproduct $\Delta : A \to A \otimes A$ and an algebra homomorphism $A \to A^{\text{op}}$ satisfying various compatibility conditions.

Let G be a semisimple Lie group with Lie algebra $\mathfrak{g}$. Recall that the universal enveloping algebra $U(\mathfrak{g})$ is the free associative algebra generated by elements of $\mathfrak{g}$, modulo the relations $xy - yx = [x, y]$ for $x, y \in \mathfrak{g}$. In fact, $U(\mathfrak{g})$ is an example of a Hopf algebra, with $\Delta(x) := x \otimes 1 + 1 \otimes x$ and $S(x) := -x$.

Now the idea is to study deformations of G by deforming $U(\mathfrak{g})$ as a Hopf algebra. In the case at hand, viewing $U(\mathfrak{g})$ as an associative algebra we have:

Proposition 7.2. $HH^2(U(\mathfrak{g}), U(\mathfrak{g})) = 0$.

This actually implies that all formal deformations of A (i.e. over $\mathbb{K}[[h]]$) do not change $U(\mathfrak{g})$ as an associative algebra. However, there can still be interesting deformations of $U(\mathfrak{g})$ as a Hopf algebra. Recall that the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ is generated by variables X, Y, H, with Lie brackets

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

The quantum group $U_h(\mathfrak{sl}_2\mathbb{C})$ is by definition the Hopf algebra with generators X, Y, H, with:

$$[H,X] = 2X, \quad [H,Y] = -2Y, \quad [X,Y] = \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}$$
$$S(H) = -H, \quad S(X) = -Xe^{-hH}, \quad S(Y) = -e^{hH}Y$$
$$\Delta(X) = X \otimes e^{hH} + 1 \otimes X, \quad \Delta(Y) = Y \otimes 1 + e^{-hH} \otimes Y,$$
$$\Delta(H) = H \otimes 1 + 1 \otimes H.$$

8. Deforming Poisson manifolds

Let M be a smooth manifold, and set $A = C^\infty(M)$.

Definition 8.1. A smooth deformation of A is an associative product $* : A \otimes A \to A[[h]]$ of the form $f * g = fg + \sum_{k=1}^\infty c_k(f,g)h^k$, such that

- $c_1(f,g) - c_1(g,f) = 2\{f,g\}$
- each $c_k : A \times A \to A$ is a bidifferential operator.

The latter condition means that in local coordinates c_k is of the form $f \otimes g \mapsto \sum_{I,J} Q_{I,J}(x)\partial^I(f)\partial^J(g)$, where I, J are multi-indices and $Q_{I,J}$ is a smooth function.

We now consider two DGLAs. Firstly, let $D^*(X)$ denote the subcomplex of $CC^*(A,A)$ consisting of Hochschild cochains which are polydifferential operators. Secondly, let $T^*(X) = \Gamma(X, \Lambda^* TX)$ denote polyvector fields on X, equipped with the trivial differential and the "Schouten–Nijenhuis" bracket (this extends the usual bracket of holomorphic vector fields).

Proposition 8.2. *Maurer–Cartan elements modulo gauge equivalence in $T^*(X)$ are in one-to-one correspondence with Poisson structures on X modulo diffeomorphism.*

Theorem 8.3. *(Kontsevich) There is a quasi-isomorphism $T^*(X) \simeq D^*(X)$.*

It moreover follows that the sets of Maurer–Cartan elements modulo gauge equivalence coincide for both sides. Since $D^*(X)$ governs smooth deformations of A, this shows that every Poisson manifold can be canonically quantized, resolving the question from the beginning.

References

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