Bott Periodicity and Clifford Algebras

Kyler Siegel

November 27, 2012

Contents

1 Introduction

Recall for that a map $f : S^1 \to \mathbb{C} \setminus \{0\}$ the homotopy class is uniquely characterized by an integer called the "winding number". There are many ways to compute this number, for example

- $\frac{1}{2\pi i}\int \frac{df}{f}$
- counting the algebraic number of points in the pre-image of a regular value of $f/|f|$ (essentially counting the number of times the curve crosses say the x-axis).

Recall that, for a topological space X, $\pi(X) = [S^i, X]$ denotes the homotopy classes of pointed maps from S^i (the i-dimensional sphere) to X. So the point here is that $\pi_1(\mathbb{C}\setminus\{0\}) = \mathbb{Z}$.

To generalize the winding number to higher dimensions, we might ask "what is $\pi_i(\mathrm{GL}(n,\mathbb{C}))$?". What about $\pi_i(\mathrm{GL}(n,\mathbb{R}))$? Note that we have a deformation retration

$$\mathbb{C} \setminus \{0\} \simeq S^1.$$

In fact, by a general theorem of Iwasawa, any Lie group deformation retracts onto a maximal compact subgroup (think of this as a fancier version of the above). In particular, we have homotopy equivalences

$$\begin{aligned} \mathrm{GL}(n,\mathbb{C}) &\simeq U(n) \\ \mathrm{GL}(n,\mathbb{R}) &\simeq O(n). \end{aligned}$$

Remark 1.1 *In this talk we'll focus on the real version of Bott periodicity, which involves periods of length* 8*. There is also a complex version involving periods of length* 2*, which is often simpler.*

So we have isomorphisms $\pi(\mathrm{GL}(n,\mathbb{R})) \cong \pi_i(O(n))$. Unfortunately, in general homotopy groups can be extremely mysterious. For example, $\pi_i(S^n)$ is one of the first computations one would naturally ask about, and these are completely unknown in general. Similarly, $\pi_i(O(n))$ is unknown in general.

However, recall that there are inclusions $O(n) \subset O(n+1)$ given by sending $A \in O(n)$ to the block matrix

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

It's easy to show that the induced map $\pi_i(O(n)) \to \pi_i(O(n+1))$ is an isomorphism for n sufficiently large. Setting

$$O = \cup_n O(n),$$

we have $\pi_i(O) = \pi_i(O(n))$ for n sufficiently large.

In 1957, Bott calculated these "stable" homotopy groups using infinite dimensional Morse theory. The result:

i	$\pi_i(O)$
0	$\mathbb{Z}/2$
1	$\mathbb{Z}/2$
2	0
3	$\mathbb{Z}$
4	0
5	0
6	0
7	$\mathbb{Z}$
$8j+k$	$\pi_k(O)$

The fact that the answer is periodic with period 8 is completely surprising and has repercussions throughout many parts of mathematics.

2 Clifford Algebras

Given how revolutionary the extension from $\mathbb{R}$ to $\mathbb{C}$ was, people were eager to see if $\mathbb{C}$ could be further extended to new types of numbers. Since $\dim_{\mathbb{R}} \mathbb{C} = 2$, it was natural to look for a three dimensional algebra over $\mathbb{R}$. Ideally it would have nice properties, like be a *division algebra* (meaning nonzero elements have inverses).

Famously, after failing for a long time for find such an object, in 1843 William Hamilton realized that something interesting exists in *four* dimensions. We now call these the *quaternions* (denoted by $\mathbb{H}$ in honor of Hamilton):

$$\mathbb{H} = \mathbb{R}\langle e_1, e_2 \mid e_1 e_2 = -e_2 e_1,\ e_1^2 = e_2^2 = -1\rangle.$$

The quaternions are an example of an (associative) normed division algebra over $\mathbb{R}$.

Remark 2.1 *Quaternions are* not *commutative!*

It turns out that besides $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$, there is only one other normed division algebra over $\mathbb{R}$, the *octonions* $\mathbb{O}$, discovered shortly afterwards by John Graves and Arthur Cayley independently.

- $\dim_{\mathbb{R}} \mathbb{O} = 16$
- The octonions are neither commutative nor associative!

Observe that $\mathbb{H}$ is obtained from $\mathbb{R}$ by adjoining two anticommuting square roots of -1. In 1876, William Clifford generalized $\mathbb{H}$ in a different way from $\mathbb{O}$ by defining the *Clifford algebra*:

$$C_k = \mathbb{R}\langle e_1, ..., e_k \mid e_i e_j = -e_j e_i \text{ for } i \neq j,\ e_i^2 = -1\rangle.$$

Equivalently, let

$$T(\mathbb{R}^k) = \bigoplus_{i=0}^{\infty} T^i(\mathbb{R}^k) = (\mathbb{R}) \oplus (\mathbb{R}^k) \oplus (\mathbb{R}^k \otimes \mathbb{R}^k) \oplus ...$$

be the tensor algebra over $\mathbb{R}^k$, and let Q_k be the negative definite quadratic form on $\mathbb{R}^k$, given by

$$Q_k(x_1, ..., x_k) = -x_1^2 - ... - x_k^2.$$

Then the Clifford algebra is a quotient

$$C_k = C(Q_k) := T(\mathbb{R}^k)/\langle x \otimes x - Q_k(x) \cdot 1 \mid x \in \mathbb{R}^k\rangle,$$

where $\langle x \otimes x - Q_k(x) \cdot 1 \mid x \in \mathbb{R}^k\rangle$ denotes the two-sided ideal generated by elements of the form $x \otimes x - Q_k(x) \cdot 1$.

We make the following observations:

- This construction works equally well for any quadratic form Q, giving rise a Clifford algebra $C(Q)$. The choice $Q = Q_k$ corresponds to adding square roots of -1.

- For $Q \equiv 0$, we have $C_k(Q) = \Lambda(\mathbb{R}^k)$, the exterior algebra over $\mathbb{R}^k$. In fact, we always have

$$\dim_R(C(Q)) = \dim_{\mathbb{R}} \Lambda(\mathbb{R}^k) = 2^k,$$

  with a basis given by $\{e_{i_1}e_{i_2}...e_{i_k},\ i_1 < i_2 < ... < i_k\}$ for $\{e_i\} \subset \mathbb{R}^k$ any basis.

- $C(Q)$ is characterized by the following universal property:

  For any linear map $\phi : \mathbb{R}^k \to A$ into an $\mathbb{R}$-algebra with unit A, such that $\forall x \in \mathbb{R}^k$ we have $\phi(x)^2 = Q(x) \cdot 1$, there exists a unique algebra homomorphism $\overline{\phi} : C(q) \to A$ such that $\overline{\phi} \circ i_Q = \phi$.

It will be useful to also define

$$C'_k := C(-Q_k) = \mathbb{R}\langle e'_1, ..., e'_k \mid e'_i e'_j = -e'_j e'_i \text{ for } i \neq j,\ e'^2_i = 1\rangle.$$

Determining the algebras C_k:
Let $F = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, and let $F(n)$ denote the $n \times n$ matrix algebra over F.

Exercise 2.2

- $F(n) \cong \mathbb{R}(n) \otimes_{\mathbb{R}} F$
- $\mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{R}(m) \cong \mathbb{R}(nm)$
- $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$
- $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2)$
- $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4)$

Proposition 2.3

- $C_k \otimes_{\mathbb{R}} C'_2 \cong C'_{k+2}$
- $C'_k \otimes_{\mathbb{R}} C_2 \cong C_{k+2}$

Proof Define $\psi : (\mathbb{R}')^{k+2} \to C_k \otimes_{\mathbb{R}} C'_2$ by

$$\psi(e'_i) = \begin{cases} e_{i-2} \otimes e'_1 e'_2 & : 2 < i \leq k \\ 1 \otimes e'_i & : 1 \leq i \leq 2 \end{cases}$$

We compute:

for $2 < i \leq k$,

$$\psi(e_i')^2 = e_{i-2}^2 \otimes e_1' e_2' e_1' e_2' = (-Q_k)(e_i')(1 \otimes 1)$$

for $1 \leq i \leq 2$,

$$\psi(e_i')^2 = (1 \otimes e_i')^2 = (-Q_k)(e_i')(1 \otimes 1).$$

By the universal property of C'_{k+2}, we get an algebra homomorphism $C'_{k+2} \to C_k \otimes_{\mathbb{R}} C'_2$. By observing what it does on bases, this is easily seen to be an isomorphism. Similarly, we get an isomorphism $C_{k+2} \cong C'_k \otimes_{\mathbb{R}} C_2$.

Exercise 2.4

- $C_1 \cong \mathbb{C}$
- $C'_1 \cong \mathbb{R} \oplus \mathbb{R}$
- $C_2 \cong \mathbb{H}$
- $C'_2 \cong \mathbb{R}(2)$.

Using the above ingredients, we can easily fill out the table:

k	C_k	C'_k	Irreps of C_k
0	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{C}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{C}$
2	$\mathbb{H}$	$\mathbb{R}(2)$	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{C}(2)$	$\mathbb{H}_-, \mathbb{H}_+$
4	$\mathbb{H}(2)$	$\mathbb{H}(2)$	$\mathbb{H}^2$
5	$\mathbb{C}(4)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{C}^4$
6	$\mathbb{R}(8)$	$\mathbb{H}(4)$	$\mathbb{R}^8$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{R}^8_-, \mathbb{R}^8_+$
8	$\mathbb{R}(16)$	$\mathbb{R}(16)$	$\mathbb{R}^{16}$
$8j + k$	$C_k \otimes_{\mathbb{R}} \mathbb{R}(16^j)$	$C'_k \otimes_{\mathbb{R}} \mathbb{R}(16^j)$	

Remark 2.5 *For the last column we are using the standard fact that the matrix ring $F(n)$ admits only the obvious irreducible representation, namely the represetation F^n on which $F(n)$ acts by matrix multiplication.*

For example, we have

$$\begin{aligned} C_{k+8} &\cong C_k \otimes_{\mathbb{R}} C'_2 \otimes_{\mathbb{R}} C_2 \otimes_{\mathbb{R}} C'_2 \otimes_{\mathbb{R}} C_2 \\ &\cong C_k \otimes_{\mathbb{R}} C_4 \otimes_{\mathbb{R}} C_4 \\ &\cong C_k \otimes_{\mathbb{R}} \mathbb{R}(16). \end{aligned}$$

Note that, up to tensoring with $\mathbb{R}(16^j)$, the table is 8-periodic! This is Bott periodicity!

3 Vector Fields on Spheres

Recall the "hairy ball theorem": S^2 does not admit a nonvanishing vector field. The slogan is "you can't comb a hairy ball"! Actually, the Euler characteristic χ gives the number of zeroes of a generic vector field. We have

$$\chi(S^{2n}) = 2$$
$$\chi(S^{2n+1}) = 0,$$

so this immediately shows that the hairy ball theorem is true for any even dimensional sphere S^{2n}. It is *false* for odd dimensional spheres, but *how false?*. That is, we can ask

Question 3.1 *How many linearly independent vector fields does S^{2n+1} admit?*

Remark 3.2 *If S^n admits n linearly independent vector fields, we say it is* parallelizable. *It is a deep result that this occurs only when $n = 0, 1, 3, 7$. This is intimately related to the fact that $\mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$ are the only normed division algebras over $\mathbb{R}$.*

The answer to Question 3.1 was given by Adams in 1962:

There are exactly as many linearly indepedent vectors fields on S^n as can be naturally constructed using Clifford algebras.

Remark 3.3 *We will show below how to construction vector fields on spheres using Clifford algebras. The other direction, i.e. showing that this is optimal, is much harder and was proved using the Adams operations in K theory.*

Our main tool for construction vector fields on spheres will be the following:

Proposition 3.4 *If $\mathbb{R}^n$ admits the structure of a C_k-module, then S^{n-1} admits k orthonormal vector fields.*

Proof We give just a sketch of the proof. Note that $\mathbb{R}^n$ being a C_k-module is the same as having a ring homomorphism

$$\phi : C_k \to M_n(\mathbb{R}).$$

By averaging any inner product on $\mathbb{R}^n$ over the group $\Gamma := \langle \phi(e_1), ..., \phi(e_k) \rangle$, we can assume that C_k acts orthogonally on $\mathbb{R}^n$. One can then check that, for $x \in \mathbb{R}^n$, the vectors $x, \phi(e_1)x, ..., \phi(e_k)x \in \mathbb{R}^n$ are mutually orthogonal with respect to the inner product on $\mathbb{R}^n$. Indeed, it suffices to show that

$$\langle \phi(e_i)x, x \rangle = -\langle \phi(e_i)x, x \rangle$$

for any i and

$$\langle \phi(e_i)x, \phi(e_j)x \rangle = -\langle \phi(e_i)x, \phi(e_j)x \rangle$$

for any $i \neq j$.

Definition 3.5 *We define n_k to be the smallest n such that $\mathbb{R}^n$ is a simple C_k-module (i.e. the dimension over $\mathbb{R}$ of an irreducible representation of C_k) and define $\rho(n)$ to be the largest k such that $n_k|n$ (i.e. the largest k such that $\mathbb{R}^n$ is a module over C^k).*

Since $\mathbb{R}^n$ is a $C_{\rho(n)}$-module, the above proposition shows that S^{n-1} admits $\rho(n)$ linearly independent vector fields. We can now restate Adams' theorem more formally as

Theorem 3.6 *(Adams) The maximal number of linearly independent vectors fields on S^{n-1} is precisely $\rho(n)$.*

Exercise 3.7 *For $n = 16^a 2^b m$ with m odd and $0 \leq b \leq 3$, we have*

$$\rho(n) = 8a + 2^b - 1.$$

The first few values of ρ are given by:

n	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
$\rho(n+1)$	1	3	1	7	1	3	1	8	1	3	1	7	1	3	1	9

Remark 3.8 *The $\rho(n)$'s are called "Radon-Hurwitz numbers".*

4 Division Algebras

Our main tool for studying division algebras will be

Proposition 4.1 *If $\mathbb{K}$ is an n-dimensional normed division algebra over $\mathbb{R}$, then $\mathbb{R}^n$ admits the structure of a C_{k-1}-module.*

Proof Firstly we'll need:

Exercise 4.2 *Show that the norm $||\cdot||$ on $\mathbb{K}$ is induced by an inner product $\langle\cdot,\cdot\rangle$.* Hint: *by averaging, find* some *inner product on $\mathbb{K}$ which in invariant under left multiplication by elements $k \in \mathbb{K}$ with norm 1.*

Now let

$$\text{Im}(\mathbb{K}) := \{k \in \mathbb{K} \mid \langle k, 1\rangle = 0\}.$$

Let $L_k : \mathbb{K} \to \mathbb{K}$ denote left multiplication by k. We'll show that $L_k^2 = -I$ for any $k \in \text{Im}(\mathbb{K})$ with $||k|| = 1$. It will then follow by the universal property of C_{k-1} that we have a map

$$\overline{L} : C_{n-1} \to M_n(\mathbb{R})$$

as desired.

Now for $k \in \mathrm{Im}(\mathbb{K})$ with $||k|| = 1$, set $l = \dfrac{k+1}{\sqrt{2}}$, so that $||l|| = 1$ and $L_k, L_l \in O(n)$. Then we have

$$\begin{aligned} I = L_l L_l^t &= \frac{1}{2}(L_k + I)(L_k^t + I) \\ &= \frac{1}{2}(L_k L_k^t + L_k + L_k^t + I) \\ &= I + \frac{1}{2}(L_k + L_k^t), \end{aligned}$$

and therefore $L_k = -L_k^t$ and hence $L_k^2 = (L_k)(-L_k^t) = -I$, as desired.

Theorem 4.3 *(Hurwitz) If $\mathbb{K}$ is an m-dimensional normed division algebra, then $m = 1, 2, 4, 8$.*

Proof By Theorem 4.1, we must have $n_m | m$. Looking at the table:

$k+1$	C_k	n_k
1	$\mathbb{R}$	1
2	$\mathbb{C}$	2
3	$\mathbb{H}$	4
4	$\mathbb{H} \oplus \mathbb{H}$	4
5	$\mathbb{H}(2)$	8
6	$\mathbb{C}(4)$	8
7	$\mathbb{R}(8)$	8
8	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	8
9	$\mathbb{R}(16)$	16
10	$\mathbb{C}(16)$	32

it is clear that this can only happen when $m = 1, 2, 4$, or 8.

Remark 4.4 *The above result remains valid even if we drop the assumption that $\mathbb{K}$ be normed. Also, with a little more work we can show that $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$.*

References

[Ati] MF Atiyah. Algebraic topology and elliptic operators. *Communications on Pure and Applied Mathematics* **20**(1967), 237–249.

[ABS] M.F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology* **3**(1964), 3–38.

[Bae] J.C. Baez. The octonions.(English summary). *Bull. Amer. Math. Soc.(NS)* **39**(2002), 145–205.

[Mai] T. Maienschein. Clifford algebras, division algebras, and vector fields on spheres. 2011.