The Ubiquity of ADE Classifications in Nature

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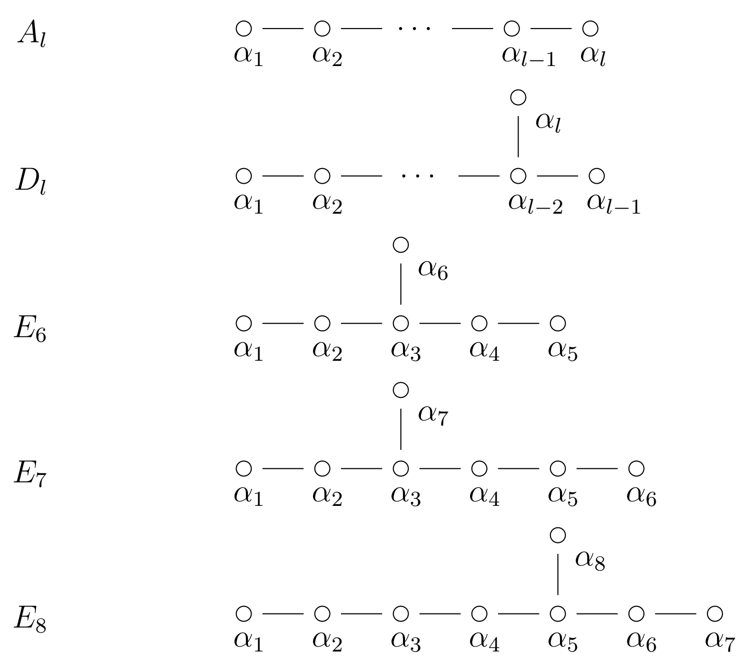
1 Introduction

The so-called "simply laced Dynkin diagrams" are the graphs A_l $(l \geq 2)$, D_l $(l \geq 4)$, E_6, E_7, and E_8. There are two regular families $\{A_l\}$ and $\{D_l\}$, and three exceptional graphs E_6, E_7, and E_8. They are given pictorially by:



It is well known that these diagrams play a key role in the classification of Lie groups and Lie algebras, along with the other Dynkin diagrams $\{B_l\}, \{C_l\}$, F_4 and G_2. However, the ADE diagrams also classify many other diverse collections of objects in mathematics and physics, many of which have seemingly nothing to do with Lie groups and Lie algebras. The diagrams arise naturally (and often very mysteriously) when studying things like quiver representations, singularity theory, linear algebra, hyper-Kahler geometry, finite subgroups of Lie groups, and even conformal field theories and string theory. It is still not entirely understood why these objects all have a common classification, although recent advances in string theory claim to shed light on this. The goal of this talk is to give a bird's eye overview of various examples of ADE classifications. It will be far from comprehensive and contain no proofs. Our primary reference is the excellent survey [HHSV].

2 The Simply Laced Dynkin Diagrams

Consider a finite connected graph Γ. We want to think of Γ as encoding a finite collection of vectors in Euclidean space, and thereby associate to Γ a reflection group generated by the orthogonal reflections about each vector. Concretely, let $W(\Gamma)$ be the "Weyl group" of Γ given as follows. $W(\Gamma)$ has one abstract generator s for each vertex s of Γ, subject to the relations $s^2 = 1$, $(ss')^2 = 1$ if there is no edge joining s and s', and $(ss')^3 = 1$ if there is an edge joining s and s'.

Theorem 2.1 *The Weyl group $W(\Gamma)$ associated to Γ is finite if and only if Γ is one the diagrams above:*

$$A_l \ (l \geq 2), \qquad D_l \ (l \geq 4), \qquad E_l \ (l = 6, 7, 8).$$

Remark 2.2 *We interpret the nodes not connected by an edge as orthgonal vectors, and vectors connected by an edge as vectors forming an angle of $2\pi/3$. The "simply laced" condition is that we're not allowing other angles.*

3 Platonic solids

The oldest ADE classification is that of the Platonic solids, known to the ancient Greeks. Recall that a Platonic solid is a polyhedron with all faces congruent regular q-gons and with p faces meeting at each vertex, for some numbers p and q. Let

$$\begin{aligned}
q &= \#\text{ of sides of each face}\\
p &= \#\text{ of faces meeting at each vertex}\\
f &= \#\text{ of faces}\\
e &= \#\text{ of edges}\\
v &= \#\text{ of vertices}
\end{aligned}$$

To classify the Platonic solids, recall that a regular q-gon has interior angle $\pi(1 - 2/q)$. Since p of them meet at a vertex, we must have

$$2\pi > p\pi(1 - 2/q).$$

$\{\mathbf{q},\mathbf{p}\}$	**name**	**f**	**e**	**v**	**symmetry group**
$\{q,2\}$	dihedron	2	q	q	$\mathbb{D}(q)$
$\{2,p\}$	hosohedron	p	p	2	$\mathbb{D}(q)$
$\{3,3\}$	tetrahedron	4	6	4	$\mathbb{T}_{12}$
$\{4,3\}$	cube	6	12	8	$\mathbb{O}_{24}$
$\{3,4\}$	octahedron	8	12	6	$\mathbb{O}_{24}$
$\{5,3\}$	dodecahedron	12	30	20	$\mathbb{I}_{60}$
$\{3,5\}$	icosahedron	20	30	12	$\mathbb{I}_{60}$

This places severe restrictions on the possibilities for p and q, which are listed in the following table. Note that we are allowing the degenerate cases of $p = 2$ and $q = 2$.

Given a Platonic solid, we can construct a dual solid by placing the new vertices at the center of each face. This gives the dualities

$$\begin{aligned} \text{dihedron} &\longleftrightarrow \text{hosohedron} \\ \text{tetrahedron} &\longleftrightarrow \text{tetrahedron} \\ \text{cube} &\longleftrightarrow \text{octahedron} \\ \text{dodecahedron} &\longleftrightarrow \text{icosahedron.} \end{aligned}$$

Notice in the table that dual polyhedra have the same symmetry groups. Here $\mathbb{D}(q)$ denotes the dihedral group of the regular q-gon, i.e. the symmetry group in which we allow reflections. $\mathbb{T}_{12}$, $\mathbb{O}_{24}$, and $\mathbb{I}_{60}$ denote the tetrahedral group, octahedral group, and icosahedral group of orders 12, 24 and 60 respectively.

If we append the cyclic group $\mathbb{Z}/(l\mathbb{Z})$ to this list, we get our ADE picture:

$$\begin{aligned} A_l &\longleftrightarrow \mathbb{Z}/(l\mathbb{Z}) \\ D_l &\longleftrightarrow \mathbb{D}(l) \\ E_6 &\longleftrightarrow \mathbb{T}_{12} \\ E_7 &\longleftrightarrow \mathbb{O}_{24} \\ E_8 &\longleftrightarrow \mathbb{I}_{60}. \end{aligned}$$

To make this correspondence more compelling, we need to see how to associate the graphs to the groups. For this we'll need *McKay Correspondence.* Let $G \subset SO(3)$ be a finite subgroup, and let $\tilde{G}$ denote the pre-image of G under the double cover $SU(2) \to SO(3)$. Let $R_1, R_2, R_3, ...$ be the irreducible complex representations of $\tilde{G}$ and let R denote the fundamental (2-dimensional) representation of $SU(2)$, restrict to $\tilde{G}$. Then each representation $R \otimes R_i$ decomposes into irreducibles as

$$R \otimes R_j = \oplus_j m_{ij} R_j$$

for some m_{ij}. Construct the *McKay graph* $\Gamma(G)$ of G with one node for each R_i and labeled edges according to the m_{ij}.

Theorem 3.1 *The assignment $G \mapsto \Gamma(G)$ sets up a one-to-one correspondence between finite subgroups of $SO(3)$ and the affine Dynkin diagrams*

$$\tilde{A}_l \ (l \geq 2), \qquad \tilde{D}_l \ (l \geq 4), \qquad \tilde{E}_l \ (l = 6, 7, 8).$$

The affine Dynkin diagrams are the same as the ADE diagrams except that we have to add one node and one edge to each.

4 Matrices over $\mathbb{N}$ with norm less than 2

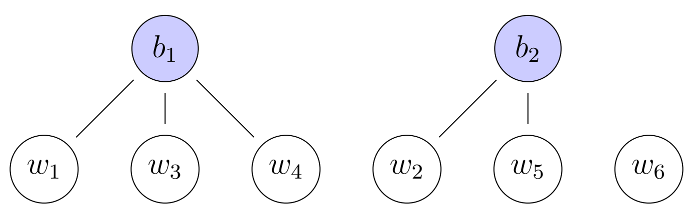
In this section we mostly follow the exposition of [GdLHJ], Chapter 1. Firstly, as motivation we have the following elementary theorem due to Kronecker:

Theorem 4.1 *(Kronecker) Let X be a finite matrix over $\mathbb{Z}$. Then if $||X|| < 2$, we have $||X|| = 2\cos(\pi/q)$ for some $q \in \{2, 3, 4, ...\}$.*

Now let $\mathrm{Mat}_{\mathrm{fin}}(\{0,1\})$ denote the set of finite matrices (not necessarily square) each entry equal to 0 or 1. We can encode an $m \times n$ matrix $X \in \mathrm{Mat}_{\mathrm{fin}}(\{0,1\})$ as a bicolored graph $\Gamma(X)$ as follows. $\Gamma(X)$ has m black vertices $b_1, ..., b_m$ and n white vertices $w_1, ..., w_n$. There are no edges joining vertices of the same color, and there is an edge joining b_i and w_j if and only if $X_{i,j} \neq 0$. For example, if

$$X = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

then $\Gamma(X)$ is given by



Definition 4.2

- *Two matrices $X_1, X_2 \in Mat_{m,n}(\mathbb{R})$ are* pseudo-equivalent *is $X_2 = P_1 X_1 P_2$ for $P_1 \in Mat_{m,m}(\mathbb{R})$ and $P_2 \in Mat_{n,n}(\mathbb{R})$ permutation matrices.*
- *A matrix $X \in Mat_{m,n}$ is* indecomposable *if it has no row or column of zeroes and it is not pseudo-equivalent to a block diagonal matrix of the form $\begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix}$ with $X' \in Mat_{m',n'}(\mathbb{R})$ and $X'' \in Mat_{m'',n''}(\mathbb{R})$, $m', n', m'', n'' \geq 1$.*

Theorem 4.3 *The encoding described above sets up a bijection between*

- *Indecomposable matrices in* $Mat_{fin}(\mathbb{N})$ *of (operator) norm less than* 2*, up to pseudo-equivalence.*

- *The following graphs, equipped with bicolorations, up to isomorphism of bicolored graphs:*

$$A_l\ (l \geq 2), \qquad D_l\ (l \geq 4), \qquad E_l\ (l = 6, 7, 8).$$

5 Quiver representations

Definition 5.1
- *A* quiver Q *is a multidigraph, i.e. a graph with directed edges where we allow repeated edges. We will alway assume* Q *is finite and connected).*

- *A* representation *of* Q *over a field* $\mathbf{k}$ *assigns to each vertex of* Q *a (finite dimensional) vector space over* $\mathbf{k}$ *and to each arrow of* Q *a linear map between the corresponding vector spaces.*

- *For* V_1, V_2 *representations of* Q*, the direct sum representation* $V_1 \oplus V_2$ *is formed taking direct sums of the corresponding vector spaces and linear maps. A quiver representation* V *is* indecomposable *if it cannot be wrriten as a direct sum of nonzero representations.*

- *Two* Q *representations are* isomorphic *if, for each vertex of* Q*, there is an isomorphism of the corresponding vector spaces, such that the natural square diagrams these form along with the edge morphisms are commutative.*

Example 5.2 *Consider the quiver with a single node and a singular directed edge (from the node to itself). A representation consists of a square matrix* M*, and two representations are isomorphic if and only if the associated matrices* M *and* M' *are conjugate (and in particular have the same dimension). Thus over an algebraically closed field* $\mathbf{k}$*, the indecomposable representations are those with* M *having a single block in its Jordan normal form, and therefore the indecomposable representations are classifed by their size and the eigenvalue appearing.*

It turns out that a quiver Q falls into one of three disjoint classes, depending on how complicated its representation theory is.

Definition 5.3
- Q *is* finite type *if, up to isomorphism, there are only finitely many indecomposable representations of* Q*.*

- Q *is* tame *if the indecomposable representations in every dimension occur in a finite number of one-parameter families.*

- Q *is* wild *if the indecomposable representations occur in families of at least two parameters.*

Example 5.4 *Consider the quiver with two nodes and three directed edges from the first node to the second. This is an example of a wild quiver.*

The ADE classification of finite type quivers is given by the following theorem of Gabriel.

Theorem 5.5 *(Gabriel) A quiver Q is of finite type if and only if its underlying undirected graph is given by one of the following Dynkin diagrams:*

$$A_l \ (l \geq 2), \quad D_l \ (l \geq 4), \quad E_l \ (l = 6, 7, 8).$$

In fact, we also have a complete understanding of the indecomposable representations of Q as follows. Let $p_1, ..., p_l$ denote the vertices of Q. For V a representation of Q, let $n(V)$ denote the vector $n(V) = (\dim V(p_1), ..., \dim V(p_l))$.

Theorem 5.6 *(Gabriel) For Q a finite type quiver, the map $V \mapsto n(V)$ sets up a bijective correspondence between indecomposable represetnations of Q and the set of positive roots of the underlying Dynkin diagram of V.*

There is also an extension to the case of tame quivers.

Theorem 5.7 *(Nazarova) A quiver Q is of tame type if and only if its underlying undirected graph is given by one of the following affine Dynkin diagrams:*

$$\tilde{A}_l \ (l \geq 2), \quad \tilde{D}_l \ (l \geq 4), \quad \tilde{E}_l \ (l = 6, 7, 8).$$

6 Simple singularities

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic map with an isolated critical point. For simplicity assume the critical point is $0 \in \mathbb{C}^n$ and we have $f(0) = 0$. Then $df(0) = 0$ but $df(z) \neq 0$ for $z \neq 0$ sufficiently close to 0.

Definition 6.1 *The critical point 0 is nondegenerate (or complex Morse) if*

$$\det \left(\frac{\partial^2 f}{\partial z_i \partial z_j}(0) \right) \neq 0.$$

Recall we have a very simple local normal form for nondegenerate critical points:

Proposition 6.2 *(Morse lemma) If 0 is a nondegenerate critical point of f, then there is a biholomorphic change of coordaintes ϕ such that*

$$f\phi(z_1, ..., z_n) = z_1^2 + ... + z_n^2.$$

In this section we are interested in the following question.

Question 6.3 *If 0 is a* degenerate *critical point of f, is there still a local normal form?*

Under suitable assumptions on the critical point, the answer is "yes"!

Definition 6.4 • *Two germs of holomorphic mappings f, g are* equivalent *if $g = f\phi$ for some biholomorphic change of coordinates ϕ.*

- *A germ f is called* simple *if there is a finite list of germs such that every small perturbation of f is equivalent to a germ from this list.*

Non-Example 6.5 *The function $f : \mathbb{C}^3 \to \mathbb{C}$, $f(x, y, z) = x^3+y^3+z^3$ is* not *simple. In fact, there is a one-parameter family of non-equivalent germs that arise from small perturbations of f.*

Here is the fundamental theorem about simple singularities:

Theorem 6.6 *(Arnold) The germ of f near 0 is simple if and only if f is equivalent to a germ in the following list:*

$$\begin{array}{rcc} x^{k+1} + y^2 + z_3^2 + ... + z_n^2 & \textit{type } A_k & (k \geq 0) \\ x^2y + y^{k-1} + z_3^2 + ... + z_n^2 & \textit{type } D_k & (k \geq 0) \\ x^3 + y^4 + z_3^2 + ... + z_n^2 & \textit{type } E_6 & (k \geq 0) \\ x^3 + xy^3 + z_3^2 + ... + z_n^2 & \textit{type } E_7 & (k \geq 0) \\ x^3 + y^5 + z_3^2 + ... + z_n^2 & \textit{type } E_8 & (k \geq 0) \end{array}$$

Now again let f be a holomorphic function $\mathbb{C}^n \to \mathbb{C}$ with an isolated critical point at 0 with $f(0) = 0$, and fix a small ball $B \subset \mathbb{C}^n$ around 0 and let $V^{2n-2} = f^{-1}(t) \cap B$ for some $t \in \mathbb{C}$ sufficiently small.

Theorem 6.7 *(Milnor) V^{2n-2} is homotopy equivalent to a wedge of μ spheres S^{n-1}. In particular, $H_{n-1}(V^{2n-2}, \mathbb{Z}) \cong \mathbb{Z}^\mu$ is equipped with an integral bilinear intersection form.*

After adding variables and a corresponding non-degenerate quadratic form to f (this process is called *stabilization*), we can assume $n \equiv 3 \pmod 4$. Now we consider the symmetric intersection form Q on $H_{n-1}(V^{2n-2}, \mathbb{Z})$.

Theorem 6.8 *(Tjurina) Q is negative definite if and only if the singularity of f at 0 is simple.*

It is convenient to encode the intersection form Q on $H_{n-1}(V^{2n-2}, \mathbb{Z})$ as a labelled graph $\Gamma(G)$ as follows. The nodes of the diagram correspond to "vanishing cycles", i.e. elements of $H_{n-1}(V_{2n-2}, \mathbb{Z})$ with square -2 (it turns out these form a basis for H_{n-1}). We join two nodes by an edge labelled with k if the intersection number between the corresponding vectors is equal to k.

Theorem 6.9 *The intersection form graphs $\Gamma(G)$ of the simple singularities A_k, D_k, E_k (as listed in Theorem 6.6) are given precisely by relevent corresponding Dynkin diagrams.*

We can always *Morsify* the critical point of f at 0, i.e. perturb it slightly such that the isolated critical point becomes μ (complex) Morse critical points inside U with different critical values. If we remove the critical values from $\mathbb{C}$, then we can speak of the monodromy as we travel around loops in the resulting puncutured $\mathbb{C}$. That is, we can look at how a fixed regular fiber V is mapped to itself as we travel along a path in $\mathbb{C}$ avoiding the critical points. The group of all such maps (up to isotopy) is the *monodromy group of the singularity.*

Theorem 6.10 *The following are equivalent:*

1. *the singularity of f is simple*
2. *the monodromy group of the singularity is finite*
3. *the monodromy group of the singularity is isomorphic to the corresponding Weyl group of type A_l, D_l, E_6, E_7, or E_8 (see Section 2).*

Theorem 6.11 *If two simple singularities have isomorphic monodromy groups then they are (stably) equivalent.*

To give a sense of how common or rare simple singularities are:

Theorem 6.12 *The set of all nonsimple germs of functions of $n \geq 3$ variables has codimension* 6*.*

So every $s < 6$ parameter family of functions can be generically perturbed such that all the functions have simple singularities.

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