Nagata, Segre-Harbourne-Gimigliano-Hirschowitz, Degenerations

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Outline

General setup

- Fix n general points $\{p_i = (x_i, y_i)\}$ in $\mathbb{P}^2$ and integers $\{m_i\}$.
- Polynomial f of degree d: f has multiplicity m at p: f and all partial derivatives up to order $m-1$ vanish at p
- What is the dimension of the space of polynomials f of degree d with multiplicity at least m_i at p_i for each i?

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- What is the dimension of the space of polynomials f of degree d with multiplicity at least m_i at p_i for each i?
- Not known in general. Known for small n (≤ 9). Known for low d. Known for low m_i.
- What is the lowest degree d for which such a (nonzero) polynomial exists?
- Not known....

Dimension Counts

Let $\mathcal{L}$ be this linear system: $|d; m_1, \ldots, m_n|$ (These are the coordinates in the Picard Group of the blowup: Pic $\cong \mathbb{Z}^{n+1}$.

- Virtual Dimension:

$$\text{vdim}(\mathcal{L}) = \frac{d(d+3)}{2} - \sum_i \frac{m_i(m_i+1)}{2}$$

- If $\text{vdim}(\mathcal{L}) \geq 0$, then a curve exists.
- This number can be negative, so we define the *Expected Dimension*:

$$\text{expdim}(\mathcal{L}) = \max\{\text{vdim}(\mathcal{L}), -1\}$$

- And of course we have the *actual dimension*: $\dim(\mathcal{L})$.

The Intersection Form

For $\mathcal{L} = |d; m_1, \ldots, m_n|$ or $C \in \mathcal{L}$:

- Self-intersection: $\mathcal{L}^2 = C^2 = d^2 - \sum_i m_i^2$.
- $K = | -3; -1, -1, \ldots, -1)|$ the 'canonical class' (or canonical system)
- $-K = |3; 1^n|$ the *anticanonical system*
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- $C \cdot (-K) = 3d - \sum_i m_i$
- Virtual dimension: If $\mathrm{vdim}(\mathcal{L}) = (C^2 - C \cdot K)/2$.
- (Note: $C^2 - C \cdot K$ is always an even number.)
- $\mathrm{vdim}(\mathcal{L}) \leq \mathrm{expdim}(\mathcal{L}) \leq \dim(\mathcal{L})$
- *Naive Conjecture:* $\dim(\mathcal{L}) = \mathrm{expdim}(\mathcal{L})$.
- This is false, due to the presence of....

(-1)-curves

- An irreducible curve E is a (-1)-curve if $E^2 = E\dot{K} = -1$.
- Example: A line through two points.
- Example: A conic through five points.
- Numerically, $E^2 - EK = 0$ so the curve will exist.
- Let $C = 2E$. Then $C^2 = -4$ and $C \cdot K = -2$ so $C^2 - C \cdot K = -2$.
- Hence $\text{vdim}(calL) = \text{expdim}(\mathcal{L}) = -1$: The curve $2E$ should not exist.
- But it does!

Segre's Conjecture

Conjecture

If $\mathcal{L}$ *contains an irreducible curve, then* $\dim(\mathcal{L}) = \text{expdim}(\mathcal{L})$.

This is not known for $n \geq 10$.

SHGH Conjecture

A refinement was developed: the Segre-Harbourne-Gimigliano-Hirschowitz Conjecture:

Conjecture

If $\dim(\mathcal{L}) > \operatorname{expdim}(\mathcal{L})$, *then there exists a* (-1)*-curve* E *such that* $C \cdot E \leq -2$.

This implies that if g is the equation of the (-1)-curve E, then g^2 is a factor of every polynomial defining elements of $\mathcal{L}$.
We say "$2E$ splits off $\mathcal{L}$"
This is not known for $n \geq 10$.

Nagata's Conjecture

In the development of the theory that led to Nagata's approach to Hilbert's 14-th problem, he made the following

Conjecture (Nagata's Conjecture)

If $n \geq 10$, and $\sqrt{n}d \leq \sum_i m_i$, then $\mathcal{L}$ is empty.

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Conjecture (Nagata for uniform multiplicities)

If $n \geq 10$ and all multiplicities $m_i = m$, and if $d \leq \sqrt{n}m$, then $\mathcal{L}$ is empty.

It is not known for any $n \geq 10$, which is not a perfect square.

Degeneration Methods

- A promising way to attack these problems: by using a degeneration of the problem.
- By general principles of semi-continuity, the dimension of a degenerate solution is at least the dimension of the general solution.
- SO: if the degenerate configuration has the expected dimension, then so must the general situation:

$$\mathrm{expdim}(\mathcal{L}) \leq \dim(\mathcal{L}) \leq \dim(\mathcal{L}_0) = (?)\,\mathrm{expdim}(\mathcal{L})$$

- There are many ways to make degenerations of the problem.

Horace Method

- Hirschowitz (1980's) developed the *Horace Method* for computing $\dim(\mathcal{L})$ in certain cases.
- The idea: put the points in some special position so that some irreducible curve splits off (because it will will intersect the system in too many points).
- Analyze the residual system
- Use Induction
- Successful for low multiplicities. Famous Theorem for $m = 2$ (even in higher dimensions).

Combinatorial Approaches

- This is best illustrated with $m = 2$, double points.
- Curves of degree d: hyperplane sections of the Veronese V
- Curves of degree d with a double point at p: the hyperplane must be tangent to V at p.
- So if you want double points at $p_1, \ldots, p_n$, then you look for a hyperplane tangent to V at all of these points.
- Degenerate the Veronese to a union of d^2 planes, meeting only at the coordinate points of the ambient space (which has dimension $(d+1)(d+2)/2$).
- Degenerate the points to go to n points, at most one on a plane.
- A hyperplane tangent to V_0 at one of these points must contain the plane.
- If the chosen n planes where you put the points are all disjoint, then the system will have the right dimension.

Degenerating the plane

- Suppose that you find a flat family $\mathcal{X} \to \Delta$ such that general fiber X_t $(t \neq 0)$ is an n-fold blowup of the plane, but the special fiber $X_0 = \sum V_i$ is a divisor with local normal crossings, reduced, etc.
- One can generalize the combinatorial approach and look for applications of semicontinuity.
- I.e., to show $\mathcal{L}_t$ is empty, show that the limit linear system is empty.
- One defines a bundle on the threefold $\mathcal{X}$, whose restriction to X_t is $\mathcal{L}_t$.
- Analyze the restriction $\mathcal{L}_0$ as a bundle on X_0.

Central Effectivity

- Easiest / crudest way:
- Build the bundle $\mathcal{L}$ on $\mathcal{X}$
- All limit bundles on $X_0 = \cup_i V_i$ are of the form $\mathcal{L} \otimes \mathcal{O}(\sum_i t_i V_i$ for "twisting" integers t_i.
- Restricting these to X_0 give many limit bundles, parametrized by the $\{t_i\}$
- (Not hard to see: these are all the limit bundles.)
- Lemma: If you have a curve in the general fiber, the limit of this curve in the degenerate fiber will exist, and will give an effective divisor in each component V_i.

Corollary

If, for every collection of twists $\{t_i\}$, the line bundle restricted to some component V_j of X_0 is not effective, then $\mathcal{L}_t$ is empty.

Matching

- More effective:
- The limit bundle $\mathcal{L}_0$ on X_0 is basically defined by a bundle L_j on each component V_j
- Matching condition: the restriction of L_j on V_j and L_k on V_k to the double curve $V_j \cap V_k$ must be the same.
- Hence one can compute $H^0(X_0, \mathcal{L}_0)$ as the subspace of $\bigoplus_j H^0(V_j, L_j)$ satisfying these conditions that the sections have to match on the double curves.

Corollary

If this subspace (of sections on each component that match on the double curves) is trivial, then $\mathcal{L}_t$ is empty.

Refined Matching

- Most effective:
- For a collection of sections that agree on the double curves, it is an extra condition that this section deforms to a section on the general fiber.
- There are some (somewhat complicated) conditions, depending especially on the singularities of the curves along the double curves.
- Analyzing these carefully, one arrives at a notion of *Refined Matching Conditions*: the conditions that the sections can deform.

Corollary

If none of the sections of this subspace (of sections on each component that match on the double curves) satisfy the refined matching conditions, then $\mathcal{L}_t$ is empty.

Best approximation to Nagata

Theorem

(Ciliberto-Dumitrescu-M.-Roé) If $d \leq 117/37$*, then* $|d; m^{10}|$ *is empty.*

(We note that $117/37 = 3.162162162$ and $\sqrt{10} = 3.162277660$.)
The proof uses a refined matching approach.
(I think this was improved in the last ten years, slightly.)

Thank you!