Markov staircases (Joé Brendel)

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§1 $\mathbb{RP}^2$-ellipsoids

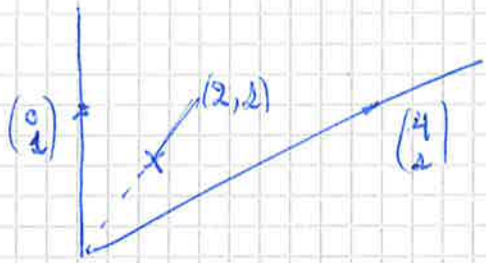
For classical staircase, look at

$\mathbb{C}^2$:

toric situation

$\mu(z_1, z_2) = \pi(|z_1|^2, |z_2|^2)$

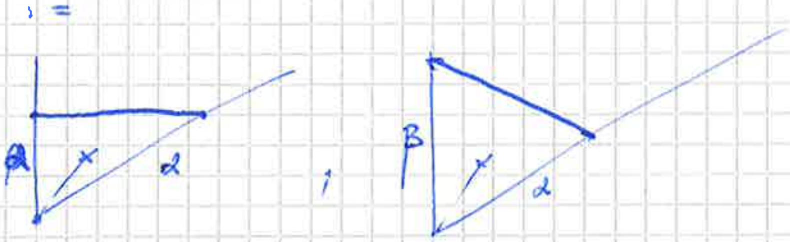
Almost toric structure on $T^*\mathbb{RP}^2$:



(honest toric)

$T^*\mathbb{RP}^2 \setminus \mathbb{RP}^2$

$E_2(\alpha,\beta)$:=



~~in partik~~

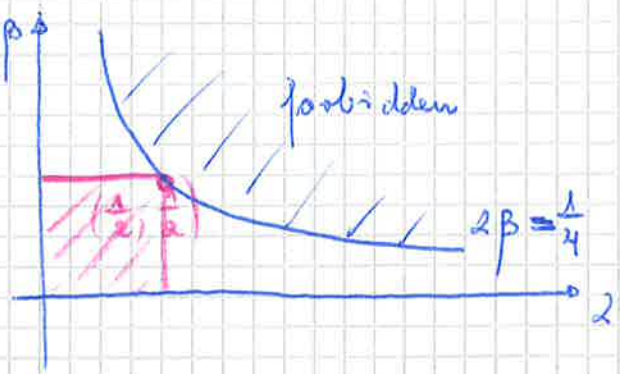
in particular: $D^*\mathbb{RP}^2 = E_2(\alpha,\alpha)$ for some α

Q. For which $(\alpha,\beta)\in\mathbb{R}^2_{>0}$:

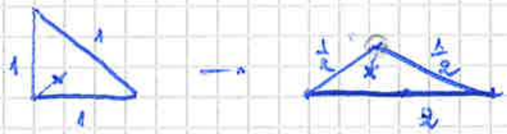
$$E_2(\alpha,\beta) \xhookrightarrow{s} \mathbb{C}P^2 := \mathbb{C}P^2(1)$$

$$\lambda_{\mathbb{R}P^2} := \text{closure of } \{(\alpha,\beta)\in\mathbb{R}^2_{>0} \mid E_2(\alpha,\beta)\xhookrightarrow{s}\mathbb{C}P^2\}$$

Answer: immediate: $\mathrm{Vol}(E_2(\alpha,\beta)) = \frac{4\alpha\beta}{2} = 2\alpha\beta \overset{!}{<} \mathrm{Vol}\,\mathbb{C}P^2(1) = \frac{1}{2}$.

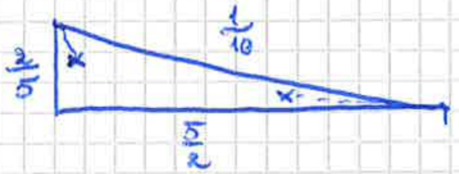


Then:



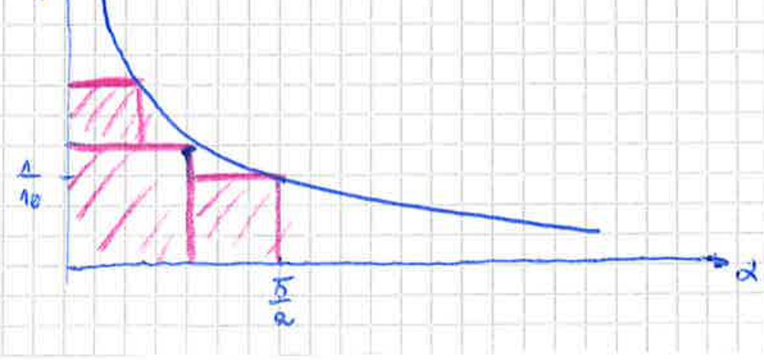
gives $E_{\mathbb{R}P^2}(\frac{1}{2},\frac{1}{2}) \hookrightarrow \mathbb{C}P^2$

Go on mutating, preserving ○ :



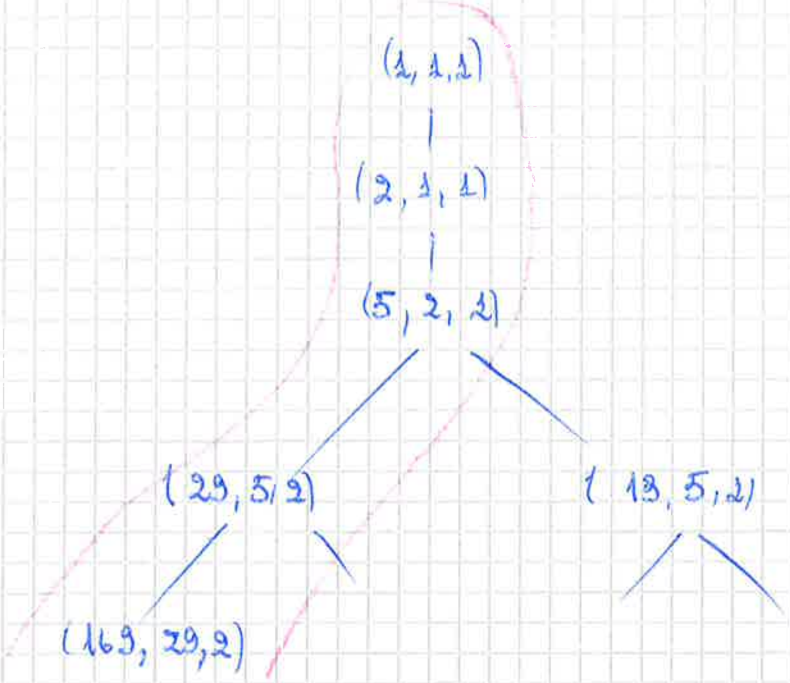
, $E_2(\frac{5}{2},\frac{1}{10}) \hookrightarrow \mathbb{C}P^2$

so:

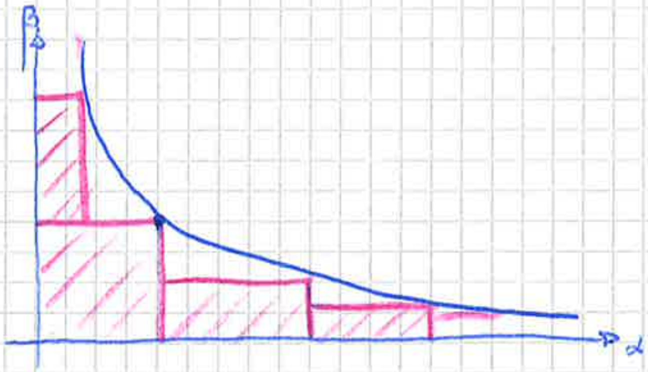


To go on, use Markov trees

solutions to $a^2+b^2+c^2 = 1$



Lemma (immediate, using ATF's).

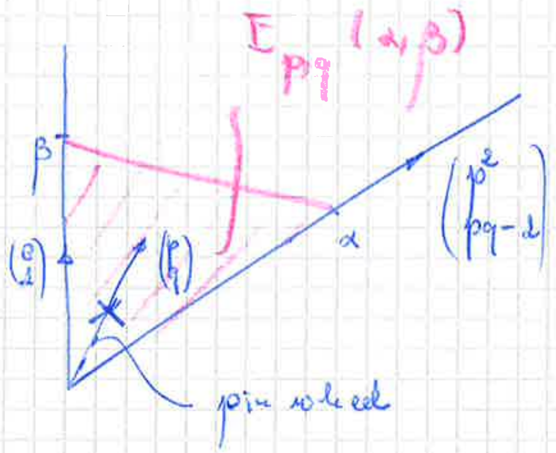


$$\bigcup_{\substack{(2,m,n)\\ \text{Markov triple}}} \left[0, \frac{n}{2m}\right] \times \left[0, \frac{m}{2n}\right] \subset \mathcal{A}_{\mathbb{R}P^2}$$

Theorem (ABEH8). $\subset$ is $=$

§2. General case

Def (Pin-ellipsoid)
(for $0 < q \le p$)
coprime

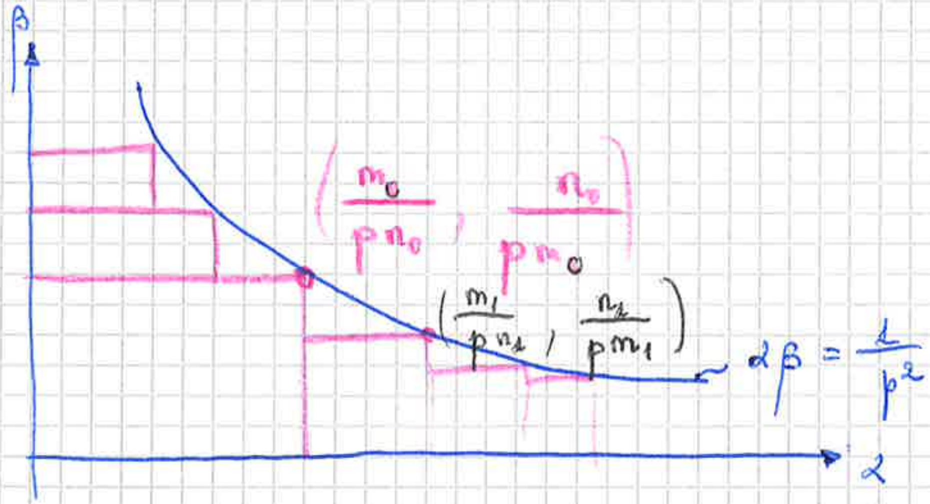


Q. For which p, q, α, β does $E_{p,q}(\alpha,\beta) \overset{s}{\hookrightarrow} \mathbb{C}P^2$?

=: admissible

Admissible (p,q) (i.e. some $E_{p,q}(\alpha,\beta) \overset{s}{\hookrightarrow} \mathbb{C}P^2$)
were determined by Evans-Smith '18 :
$q :=$ companion number of p, where
p a Markov number

Answer:



where:

(p, m_0, n_0)

(p, m_{-1}, n_{-1}) $\qquad$ (p_1, m_1, n_1)

(p, m_{-2}, n_{-2}) $\qquad$ (p_2, m_2, n_2)

Rmk. $(p,q) = (1,1)$: gives Fibonacci staircase of $E(\alpha,\beta) \xrightarrow{s} \mathbb{C}P^2$

$(p,q) = (2,1)$: gives $E_{\mathbb{R}P^2}(\alpha,\beta) \xrightarrow{s} \mathbb{C}P^2$

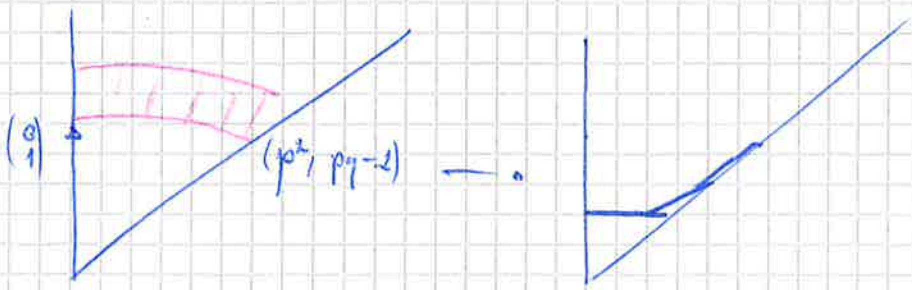
Thm (ABEHS): $\mathcal{L}_{p,q} \cap [0, \sigma_p]^2 = \bigcup_{\substack{(p,m,n) \\ \text{Markov triple}}} \left[0, \frac{m_i}{pn_i}\right] \times \left[0, \frac{n_i}{pm_i}\right]$

§3. Proof. Key idea: Rational blow-up of pinwheels / pin-ellipsoids

Examples: • $\mathbb{R}P^2 \cong L_{2,1}$: blow-up ⇝ introduce -4 curve

• $L_{5,2}$: 5 blow-ups:

toric picture:



smoothly: Fintushel - Stern,

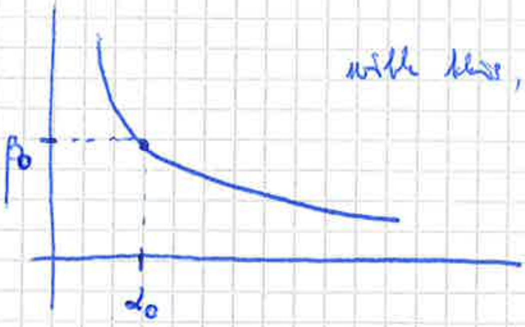
sympl.: Symington

<u>or:</u> start from orbifold:

and then do minimal resolution

This allows to see a -1 curve, that allows to see the obstruction!

<u>Thm 2</u> (ABHES): Given ι, ι_{visible} : $E_{pq}(\alpha, \beta) \xrightarrow{s} \mathbb{CP}^2$,
coming from ATF,
then exists $\varphi \in \mathrm{Ham}(\mathbb{CP}^2)$ s.t. $\iota_{vis} = \iota \circ \varphi$.



with this, get ~~Thm~~ Thm 1:

Assume $E(\alpha_0-\varepsilon, \beta_0-\varepsilon) \subset E(\lambda\alpha_0, \lambda\beta_0) \xrightarrow{s} \mathbb{CP}^2$
for $\lambda > 1$

can be assumed standard by Thm 2.

get: