The problem $\amalg_i B^4(a_i) \stackrel{s}{\hookrightarrow} B^4(\mu)$

Gromov 1985, McDuff–Polterovich 1994,
Biran 1997, Li–Liu 2001, Li–Li 2002,
Buse–Pinsonnault 2013, Karshon–Kessler 2014

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Notation

X_k : k-fold complex blow-up of $\mathbb{C}\mathrm{P}^2$

$H_2(X_k;\mathbb{Z})$ has basis $\{L, E_1, \ldots, E_k\}$

$$\ell := \mathrm{PD}(L),\ e_i := \mathrm{PD}(E_i)$$

$(d;\boldsymbol{m}) := (d; m_1, \ldots, m_k) := dL - \sum_{i=1}^k m_i E_i \in H_2(X_k;\mathbb{Z})$

$(\mu;\boldsymbol{a}) := (\mu; a_1, \ldots, a_k) := \mu\ell - \sum_{i=1}^k a_i e_i \in H^2(X_k;\mathbb{R})$

$K := -3L + \sum_{i=1}^k E_i = -\mathrm{PD}(c_1(X_k))$

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K-symplectic cone $\boxed{\mathcal{C}_K(X_k)} \subset H^2(X_k;\mathbb{R}) :=$
set of cohomology classes that can be represented by a symplectic form ω on X_k such that $c_1(\omega) = c_1(X_k) = \mathrm{PD}(-K)$

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set of exceptional classes $\boxed{\mathcal{E}_K(X_k)} \subset H_2(X_k;\mathbb{Z}) :=$
classes E with $-K \cdot E = c_1(E) = 1$, $E \cdot E = -1$ that can be represented by **smoothly embedded** spheres

By Seiberg–Witten Taubes (Li–Liu) : $\mathcal{E}_K(X_k) =$
set of classes E with $E \cdot E = -1$ that can be represented by smoothly embedded ω-**symplectic** spheres.

Theorem. The following are equivalent.

(i) There exists an embedding $\coprod_{i=1}^{k} \mathrm{B}^4(a_i) \overset{s}{\hookrightarrow} \mathrm{B}^4(\mu)$.

(ii) $\alpha := (\mu; \boldsymbol{a}) = \mu\ell - \sum_{i=1}^{k} a_i e_i \in \overline{\mathcal{C}_K}(X_k)$

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(iv) $\|\boldsymbol{a}\| \leqslant \mu$ and $a_1, \ldots, a_k \geqslant 0$, and :
$\sum_{i=1}^{k} a_i m_i \leqslant \mu d$ for every vector $(d; \boldsymbol{m}) \in \mathbb{Z}_{\geq 0}^{k+1}$ that solves the Diophantine system

$$\sum_{i=1}^{k} m_i = 3d - 1, \qquad \sum_{i=1}^{k} m_i^2 = d^2 + 1 \qquad \text{(DS)}$$

and that reduces to $(0; -1, 0, \ldots, 0)$ under repeated standard Cremona moves.

(v) $\|\boldsymbol{a}\| \leqslant \mu$, and $(\mu; \boldsymbol{a})$ reduces to a reduced vector under repeated standard Cremona moves.

Here :

For $k \geqslant 3$ define the **Cremona transform** $\mathrm{Cr}\colon \mathbb{R}^{1+k} \to \mathbb{R}^{1+k}$ as the linear map taking $(x_0; x_1, \ldots, x_k)$ to

$$(x_0 + \delta;\, x_1 + \delta,\, x_2 + \delta,\, x_3 + \delta,\, x_4,\, \ldots,\, x_k)$$

where $\boxed{\delta := x_0 - (x_1 + x_2 + x_3)}$

$(x_0; x_1, \ldots, x_k)$ is **ordered** if $x_1 \geqslant \cdots \geqslant x_k$.

standard Cremona move := order, apply Cr, order

An ordered vector $(x_0; x_1, \ldots, x_k)$ is **reduced** if $\delta \geqslant 0$ and $x_i \geqslant 0$ for all i.

A first application :

Theorem (**Packing Stability**, Biran).

For **every** $k \geq 9$, the ball $B^4(1)$ can be fully filled by k equal balls :

$$\coprod_k B^4(a) \overset{s}{\hookrightarrow} B^4(1) \quad \text{whenever the volume allows it.}$$

Proof. Let $(d; \boldsymbol{m})$ be a solution of the Diophantine system (DS).
By (DS1) in (iv) :

$$\sum_{i=1}^{k} m_i \;=\; 3d - 1 \;\leqslant\; \sqrt{k}\, d\,.$$

Take $a_i := 1$ and $\mu := \sqrt{k}$.
Then (iv) $\Rightarrow$ (i) shows that there exists an embedding

$$\coprod_{i=1}^{k} B^4(1) \overset{s}{\hookrightarrow} B^4(\sqrt{k}) \quad \square$$